Discrete Fourier Transform

P. Charistos, Prof. Ioannis Pitas
Aristotle University of Thessaloniki
pitas@csd.auth.gr
www.aiia.csd.auth.gr
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Artificial Intelligence & Information Analysis Lab

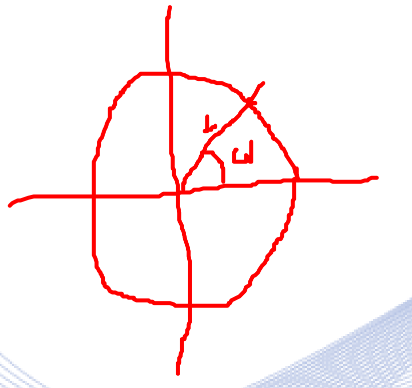
Discrete Fourier Transform (DFT)

- **Discrete-Time Fourier Transform**
- Discrete Fourier Transform
- Fast Convolution with DFT

Discrete-Time Fourier Transform

$\mathcal{Z}$ transform of a discrete signal $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$



Discrete-Time Fourier Transform (***DTFT***) is $\mathcal{Z}$ transform defined on the unit circle $z = e^{i\omega}$:

$$X\left(e^{i\omega}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{i\omega n}.$$

- The ***radial frequency*** ω is an angle defined on the unit circle: $-\pi \leq \omega \leq \pi$.

Discrete-Time Fourier Transform

It should not to be confused with analog angular frequency $\Omega = 2\pi F$. Their relation is:

$$\omega = \Omega T,$$

where T is the sampling period.

The inverse DTFT is calculated as:

$$x(n) = \frac{1}{2\pi i}\int_{z=e^{i\omega}} X(e^{i\omega})e^{i\omega(n-1)}de^{i\omega} =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\omega})e^{i\omega n}d\omega.$$

Discrete-Time Fourier Transform

Fourier transform shares many properties with $\mathcal{Z}$-transform.

Aperiodic Signal	DTFT
$x(n)$ $y(n)$	$X(\omega)$, periodic with period 2π $Y(\omega)$, periodic with period 2π
$ax(n)+by(n)$	$aX(\omega)+bY(\omega)$
$x(n-n_0)$	$e^{-i\omega n_o}X(\omega)$
$e^{j\omega_0 n}x(n)$	$X(\omega-\omega_0)$
$x^*(n)$	$X^*(-\omega)$
$x(-n)$	$X(-\omega)$
$x_{(k)}(n)=\begin{cases} x(\frac{n}{k}), & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$	$X(k\omega)$

Discrete-Time Fourier Transform

Aperiodic signal	DTFT
$x(n) * y(n)$	$X(\omega)Y(\omega)$
$x(n)y(n)$	$\frac{1}{2\pi}\int_{2\pi} X(\theta)Y(\omega-\theta)d\theta$
$x(n) - x(n) - 1$	$\left(1-e^{-i\omega}\right)X(\omega)$
$\sum_{k=-\infty}^{n} x(k)$	$\frac{1}{1-e^{-i\omega}}X(\omega) - \pi X(0)\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$
$nx(n)$	$i\frac{dX(\omega)}{d\omega}$

Discrete-Time Fourier Transform

Aperiodic signal	DTFT
real $x(n) \in \mathbb{R}$	$X(\omega) = X^*(-\omega)$ $\mathrm{Re}\{X(\omega)\} = \mathrm{Re}\{X(-\omega)\}$ $\mathrm{Im}\{X(\omega)\} = -\mathrm{Im}\{(-\omega)\}$ $\|X(\omega)\| = \|X(-\omega)\|$ $\sphericalangle X(\omega) = -\sphericalangle X(-\omega)$
real and even $x(n)$	real and even $X(\omega)$
real and odd $x(n)$	purely imaginary and odd $X(\omega)$

Discrete-Time Fourier Transform

Important DTFT properties:

- Signal time shift:

$$x(n-n_0) \leftrightarrow e^{-i\omega n_o} X(\omega).$$

- Signal convolution:

$$x(n) * y(n) \leftrightarrow X(\omega)Y(\omega).$$

- Parseval's theorem for aperiodic signals:

$$\sum_{\mathrm{n}=-\infty}^{+\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{2\pi} |\mathrm{X}(\omega)|^2 d\omega$$

Discrete-Time Fourier Transform

Discrete-time IIR system transfer function is defined by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = Hz^{N-M} \frac{\prod_{k=0}^{M}(z - c_k)}{\prod_{k=0}^{N}(z - d_k)}.$$

- $X(z)$ and $Y(z)$ are the input and output $\mathcal{Z}$-transforms.
- c_k, d_k are system zeroes and poles.
- Then its ***frequency response*** can be described by:

$$H(e^{i\omega}) = \frac{He^{i\omega(N-M)} \prod_{k=0}^{M}(e^{i\omega} - c_k)}{\prod_{k=0}^{N}(e^{i\omega} - d_k)}.$$

Discrete-Time Fourier Transform

- ***Frequency response magnitude*** is given by:

$$|H(e^{i\omega})| = \frac{|H| \prod_{k=0}^{M} |e^{i\omega} - c_k|}{\prod_{k=0}^{N} |e^{i\omega} - d_k|}.$$

- ***Frequency response phase*** is given by:

$$\phi(\omega) = (N - M)\omega + \sum_{k=1}^{M} \arg(e^{i\omega} - c_k) - \sum_{k=1}^{N} \arg(e^{i\omega} - d_k) =$$

$$= (N - M)\omega + \sum_{k=1}^{M} \tan^{-1}(\frac{\sin\omega}{\cos\omega - c_k}) - \sum_{k=1}^{N} \tan^{-1}(\frac{\sin\omega}{\cos\omega - d_k})$$

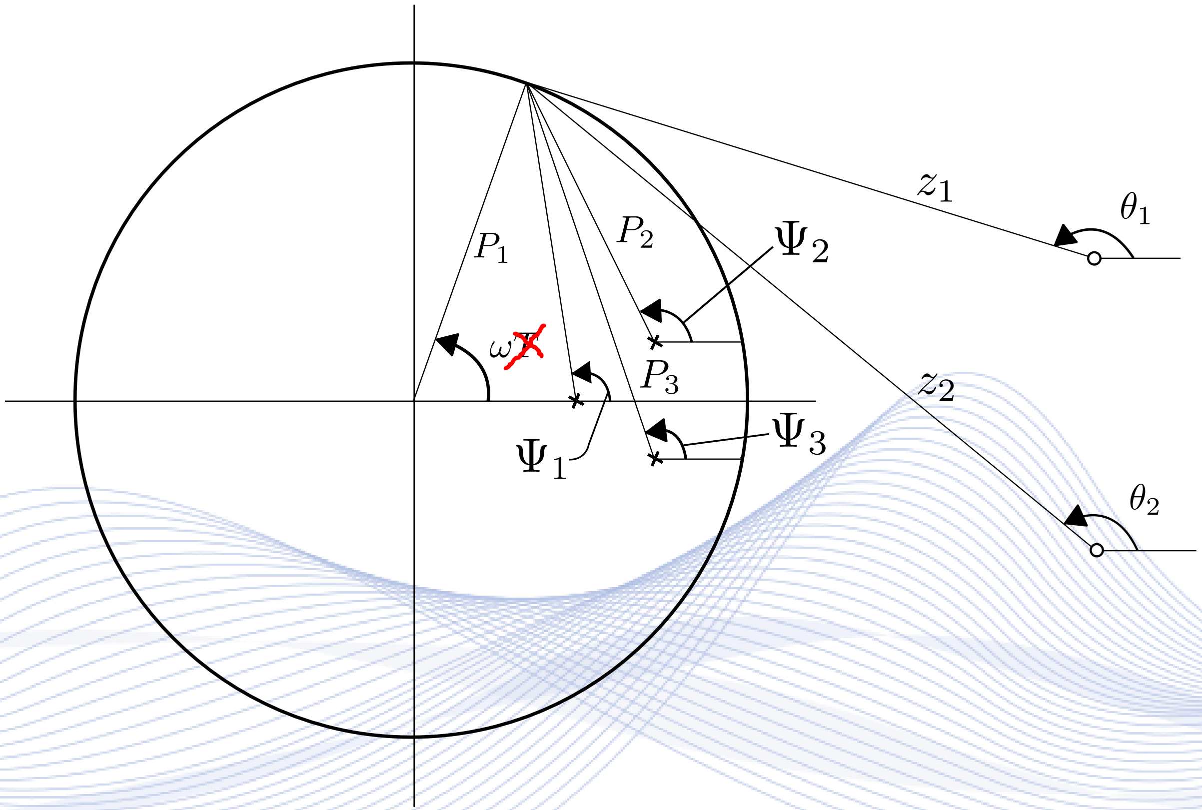
Discrete-Time Fourier Transform

If $z_k, p_k, \vartheta_k, \psi_k$ represent the magnitudes and angles of the vectors $e^{i\omega} - c_k, e^{i\omega} - d_k$, then $\left| H\left(e^{i\omega}\right) \right|$ and $\phi(\omega)$ can be expressed as:

$$\left| H\left(e^{i\omega}\right) \right| = \frac{| H | \prod_{k=0}^{M} z_k}{\prod_{k=0}^{N} p_k},$$

$$\phi(\omega) = (N - M)\omega + \sum_{k=0}^{M} \vartheta_k - \sum_{k=0}^{N} \psi_k .$$

Discrete-Time Fourier Transform



Geometric interpretation of discrete-time system frequency response.

Discrete Fourier Transform (DFT)

- Discrete-Time Fourier Transform
- **Discrete Fourier Transform**
- Fast Convolution with DFT

Discrete Fourier Transform

The ***Discrete Fourier Transform*** (***DFT***) of a sequence $x(n)$ of finite length $0 \le n \le N$, calculated in the following frequencies:

$$\omega = \frac{2\pi}{N}k, \qquad 0 \le k \le N-1,$$

is given by:

$$X(k) = X(\omega)\,|_{\omega=\frac{2\pi}{N}k} = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi nk}{N}}.$$

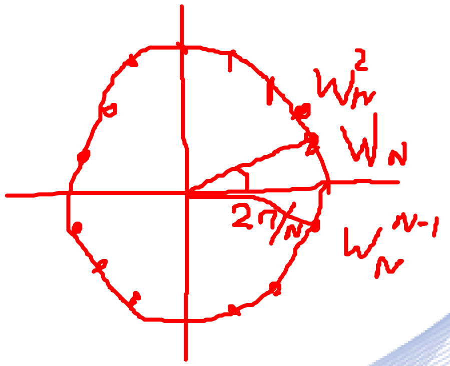
Discrete Fourier Transform

Alternative DFT notation:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

- W_N are the $N-$th complex roots of unity:

$$W_N = e^{-i\frac{2\pi}{N}}, \qquad W_N^N = 1.$$



Inverse DFT (***IDFT***) is given by:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}.$$

Inverse Discrete Fourier Transform

This can be proven by substituting $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$ in the inverse DFT definition:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} [\sum_{m=0}^{N-1} x(m) W_N^{mk}] W_N^{-nk}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x(m) \sum_{k=0}^{N-1} W_N^{k(m-n)} = \frac{1}{N} \sum_{m=0}^{N-1} x(m) \sum_{k=0}^{N-1} W_N^{k(m-n)} = x(n),$$

Inverse Discrete Fourier Transform

As:

$$\sum_{k=0}^{N-1} W_N^{k(m-n)} = \begin{cases} N, & m = n \\ 0, & m \neq n \end{cases}.$$

Discrete Fourier Transform

- DFT computation requires complex multiplications and additions.
- Each complex multiplication requires 4 real multiplications.
- DFT or IDFT computation by definition requires 2 for loops.
- Their computation complexity is $O(N^2)$.

DFT on periodic sequences

It can be proven that DFT can be defined on periodic sequences $x_p(n)$ of period N:

$$x_p(n+kN) = x_p(n).$$

The Discrete-time Fourier series is given by:

$$x_p(n) = \sum_{k=-\infty}^{\infty} X_p(k) e^{i\frac{2\pi kn}{N}}$$

DFT on periodic sequences

This relationship can be expressed for one signal period N only, since the function $e^{i\frac{2\pi kn}{N}}$ has a period N. It can, thus, take the inverse DFT form:

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_p(k)W_N^{-kn}.$$

The Discrete-time Fourier series coefficients are given by:

$$X_p(k) = \sum_{n=0}^{N-1} x_p(n)W_N^{kn}.$$

- This is the classical DFT relation pair.

Discrete Fourier Transform

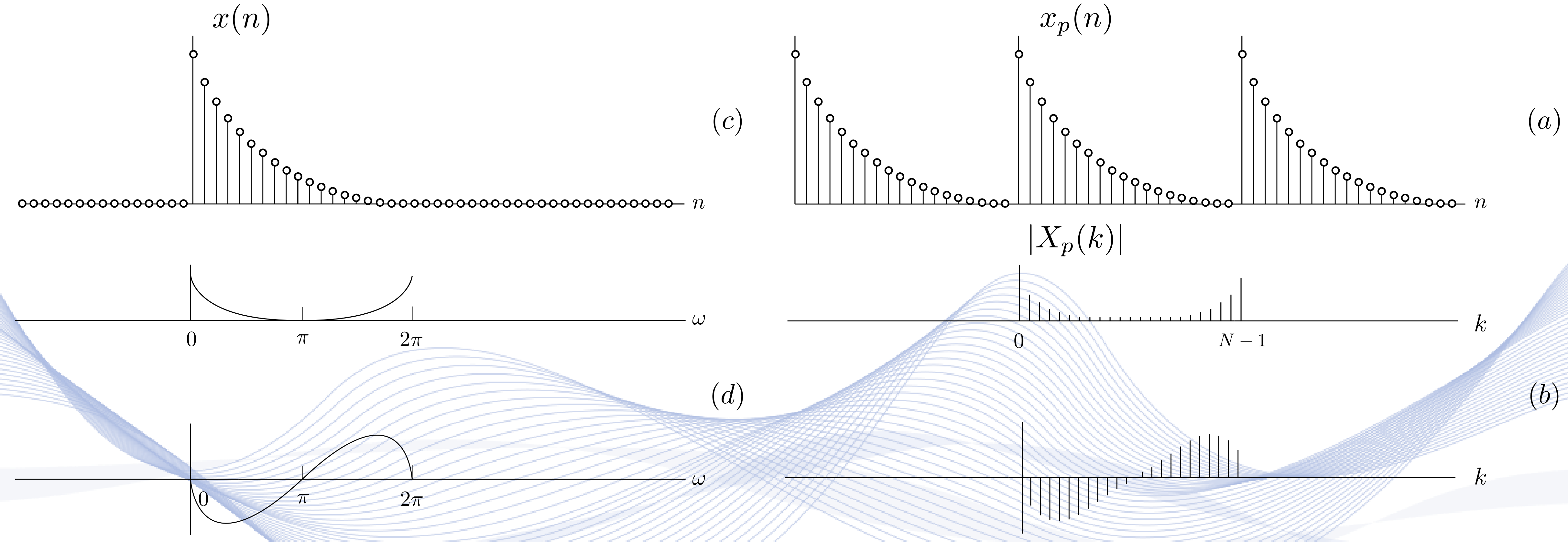
If a sequence is neither *periodic* nor of *finite length*, it **cannot** be fully, accurately represented by DFT.

This is often combatted by:

- zeroing the sequence outside the interval $[0, N-1]$.
- by periodically repeating it outside the interval $[0, N-1]$.

Then, it can be accurately described by DFT.

Discrete Fourier Transform



Aperiodic to periodic sequence extension.

Discrete Fourier Transform

DFT properties

Linearity

$$x_p(n) + y_p(n) \leftrightarrow X_p(k) + Y_p(k).$$

Periodic sequence time shift

$$x_p(n - n_o) \leftrightarrow X_p(k) W_N^{n_o k}$$

- It does not pose any difficulty, when it comes to DFT on periodic sequences.
- However, it requires further explanation concerning DFT on ***finite length sequences***.

Discrete Fourier Transform

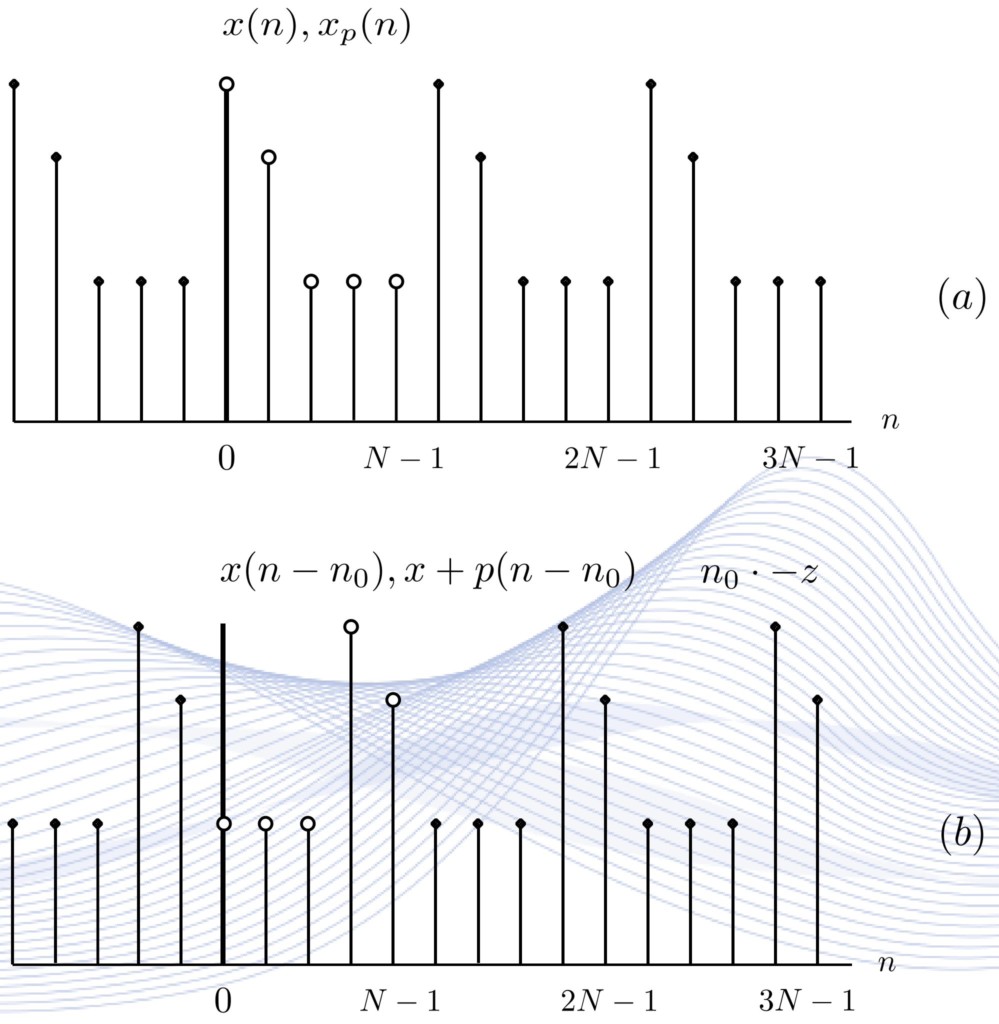
Any sequence $x(n)$ of finite length N can be periodically extended into a new sequence $x_p(n)$, whose $x(n)$ is its ***fundamental period***:

$$x(n) = \begin{cases} x_p(n), & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere.} \end{cases}$$

Circular shift: In order to shift $x(n)$, we construct $x_p(n)$ and shift $x_p(n)$ by n_o.

- Equivalently, sequence $x(n)$, $0 \leq n \leq N-1$ can be folded on a circle and shifted circularly therein.

Discrete Fourier Transform



Circular shift of a sequence $x(n)$.

Discrete Fourier Transform

DFT of real sequences.

If $x_p(n) \in \mathbb{R}$, then:

$$X_p(k) = X_p^*(N-k)$$

This property can be utilized to calculate the DFT of two real sequences $x_p(n)$, $y_p(n)$ of length N, with just one DFT of length N.

- We construct the following complex sequence of length N:

$$z_p(n) = x_p(n) + iy_p(n).$$

Discrete Fourier Transform

Its DFT:

$$Z_p(k) = \sum_{n=0}^{N-1} [x_{p(n)} + iy_p(n)] W_N^{nk}$$

can be used to calculate $X_p(k), Y_p(k)$:

$$X_p(k) = \frac{\mathrm{Re}[Z_p(k)] + \mathrm{Re}[Z_p(N-k)]}{2} + i\frac{\mathrm{Im}[Z_p(k)] - \mathrm{Im}[Z_p(N-k)]}{2}.$$

$$Y_p(k) = \frac{\mathrm{Im}[Z_p(k)] + \mathrm{Im}[Z_p(N-k)]}{2} + i\frac{\mathrm{Re}[Z_p(N-k)] - \mathrm{Re}[Z_p(k)]}{2}.$$

Discrete Fourier Transform

- The previous steps result in the calculation of the DFT of $x_p(n)$, $y_p(n)$ with less total computational cost.

- It can be as easily proven that the DFT of a real sequence $x_p(n)$ of length N can be calculated with one DFT of length $N/2$.

Discrete Fourier Transform

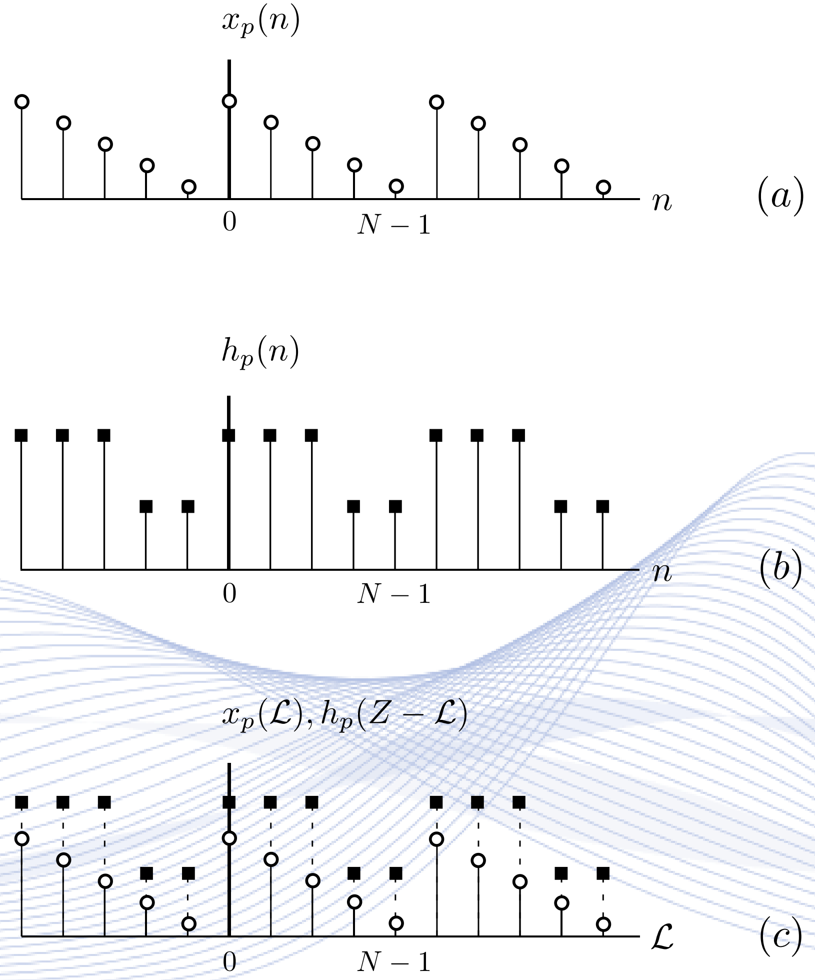
The ***cyclic convolution*** of two periodic sequences $x_p(n), h_p(n)$ is defined as:

$$y_p(n) = \sum_{l=0}^{N-1} x_p(l) h_p(n-l).$$

It can be proven that the following property applies to DFT:

$$Y_p(k) = H_p(k) X_p(k)$$

Discrete Fourier Transform



Cyclic convolution of two sequences $x_p(n), h_p(n)$.

Discrete Fourier Transform

Proof:

$$Y_p(k) = \sum_{n=0}^{N-1} [\sum_{l=0}^{N-1} x_p(l) h_p(n-l)] W_N^{lk}$$

$$= \sum_{l=0}^{N-1} x_p(l) [\sum_{n=0}^{N-1} h_p(n-l) W_N^{(n-l)k}] W_N^{lk}$$

$$= H_p(k) \sum_{l=0}^{N-1} x_p(l) W_N^{lk} = H_p(k) X_p(k).$$

Discrete Fourier Transform

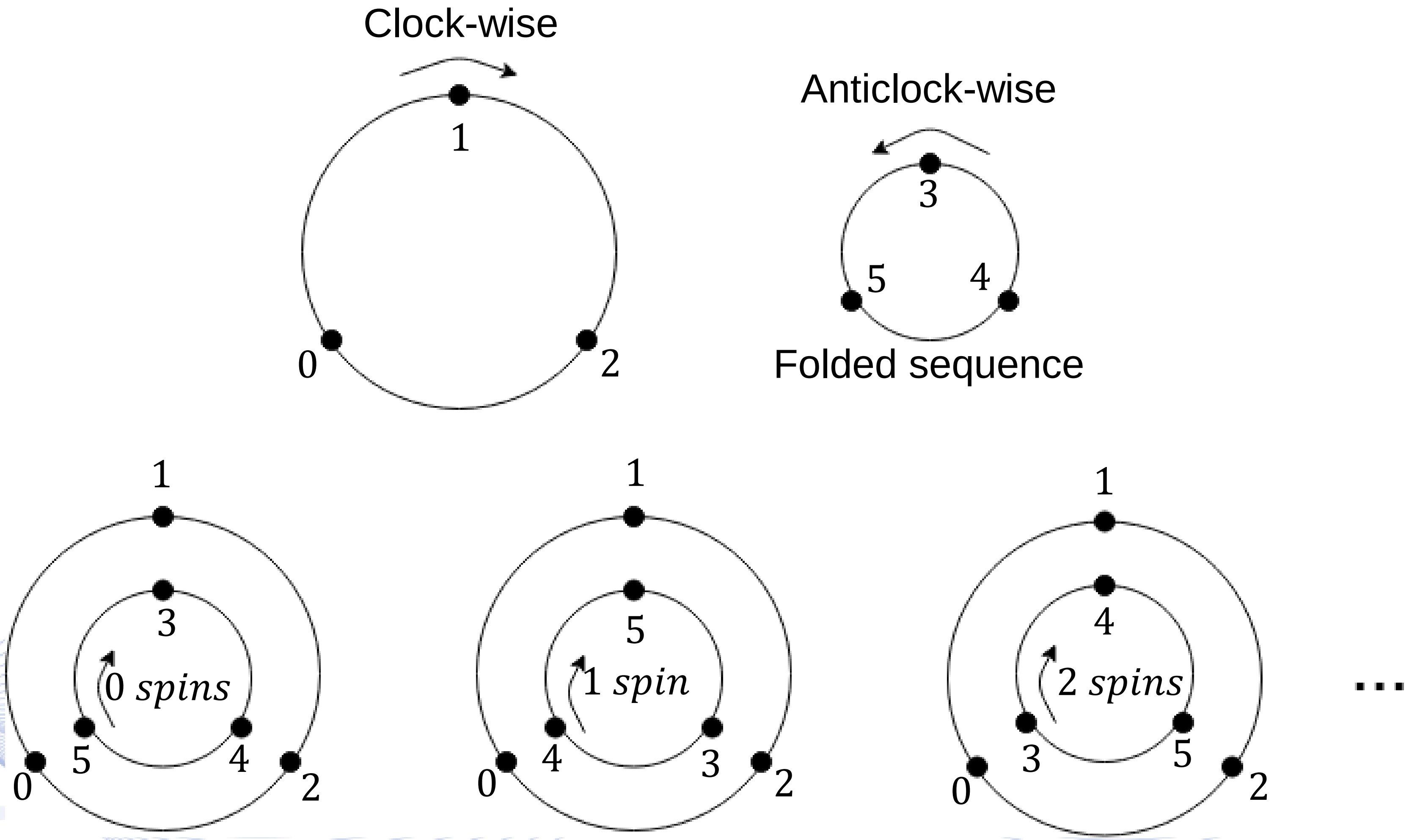
For aperiodic signals $x(k), h(k)$ of finite duration $[0, N-1]$, cyclic convolution of length N is defined as follows:

$$y(k) = x(k) \circledast h(k) = \sum_{i=0}^{N-1} h(i)x\big(((k-i)_N)\big),$$

$$(k)_N = k \mod N.$$

- It is of no much use in modeling linear systems.

Discrete Fourier Transform



$y(0) = 1 \times 3 + 2 \times 4 + 0 \times 5$ $\quad y(1) = 1 \times 5 + 2 \times 3 + 0 \times 4$ $\quad y(2) = 1 \times 4 + 2 \times 5 + 0 \times 3$

Cyclic convolution of $x(n) = \{1, 2, 0\}$ and $h(n) = \{3, 5, 4\}$.

Discrete Fourier Transform

Cyclic convolution definition as matrix-vector product:

$$\mathbf{y} = \mathbf{H}\mathbf{x},$$

- $\mathbf{x} = [x(0), \ldots, x(N-1)]^T$: the input vector.
- $\mathbf{y} = [y(0), \ldots, y(N-1)]^T$: the output vector.
- $\mathbf{H}$: a N × N ***Toeplitz matrix*** of the form:

$$\boldsymbol{H} = \begin{bmatrix} h(0) & h(1) & h(2) & \cdots & h(N-1) \\ h(N-1) & h(0) & h(1) & \cdots & h(N-2) \\ h(N-2) & h(N-1) & h(0) & \cdots & h(N-3) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h(1) & h(2) & h(3) & \cdots & h(0) \end{bmatrix}.$$

Discrete Fourier Transform (DFT)

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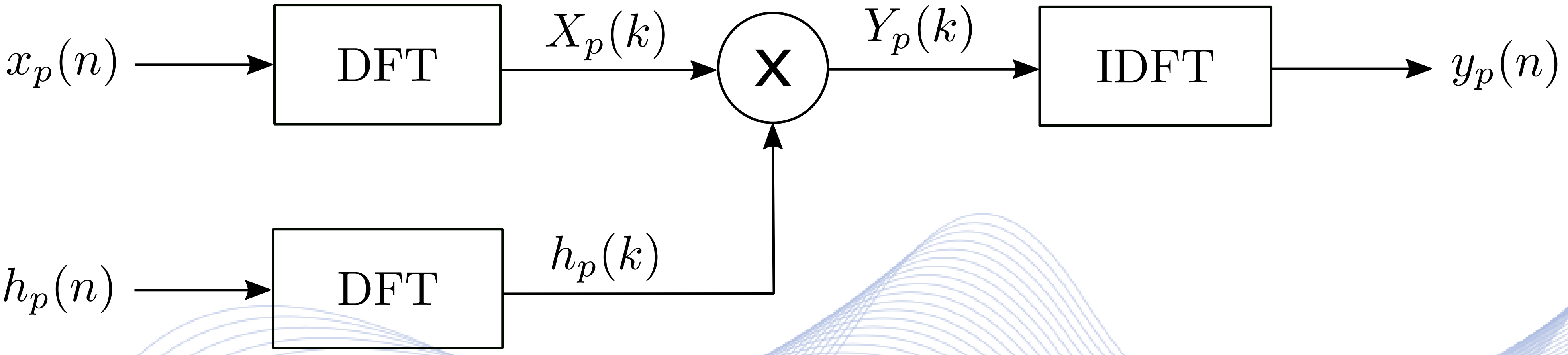
- **Fast Convolution with DFT**

Fast Convolution with DFT



Calculation of cyclic convolution via DFT.

Fast Convolution with DFT

Calculation of circular convolution via DFT can be particularly quick, since DFT can be rapidly calculated by FFT algorithms.

Linear convolution embedding in a cyclic one.

The linear convolution of two sequences $x(n), h(n)$ of lengths $L, M,$ respectively:

$$y(n) = \sum_{m=0}^{n} h(m)x(n-m)$$

has non-zero values only in the interval $[0, L+M-1]$.

Fast Convolution with DFT

Zero padding

- zeros are appended to sequences $x(n), h(n)$ in the last $L-1$, $M-1$ samples to reach a length $N \geq L+M-1$:

$$x_p(n) = \begin{cases} x(n), & 0 \leq n \leq L-1 \\ 0, & L \leq n \leq N-1 \end{cases},$$

$$h_p(n) = \begin{cases} h(n), & 0 \leq n \leq M-1 \\ 0, & M \leq n \leq N-1 \end{cases}.$$

The cyclic convolution $x_p \circledast h_p$ is calculated using the DFT.

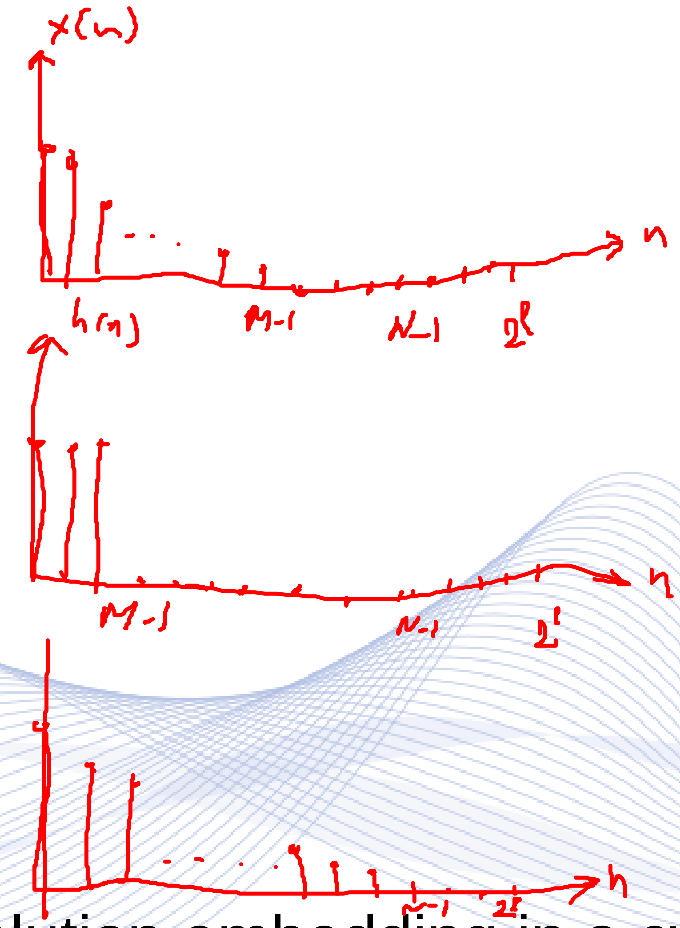
Fast Convolution with DFT

- In practice, such a periodic extension of sequences is not necessary.

- In addition, the FFT is suitable for DFT of length $L = 2^n$. Thus, the first power 2^n which surpasses $L + M - 1$ is selected as N:

$$2^n \geq N \geq L + M - 1.$$

Fast Convolution with DFT



Linear convolution embedding in a cyclic one.

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Q & A

Thank you very much for your attention!

Contact: Prof. I. Pitas
pitas@csd.auth.gr