$\mathcal{Z}$ Transform summary

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$\mathcal{Z}$ transform

- **$\mathcal{Z}$ transform**
- Inverse $\mathcal{Z}$ transform
- Properties
- Transfer Function of a Digital System
- $\mathcal{Z}$ transform and Laplace Transform

$\mathcal{Z}$ transform

$\mathcal{Z}$ transform of a discrete signal $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

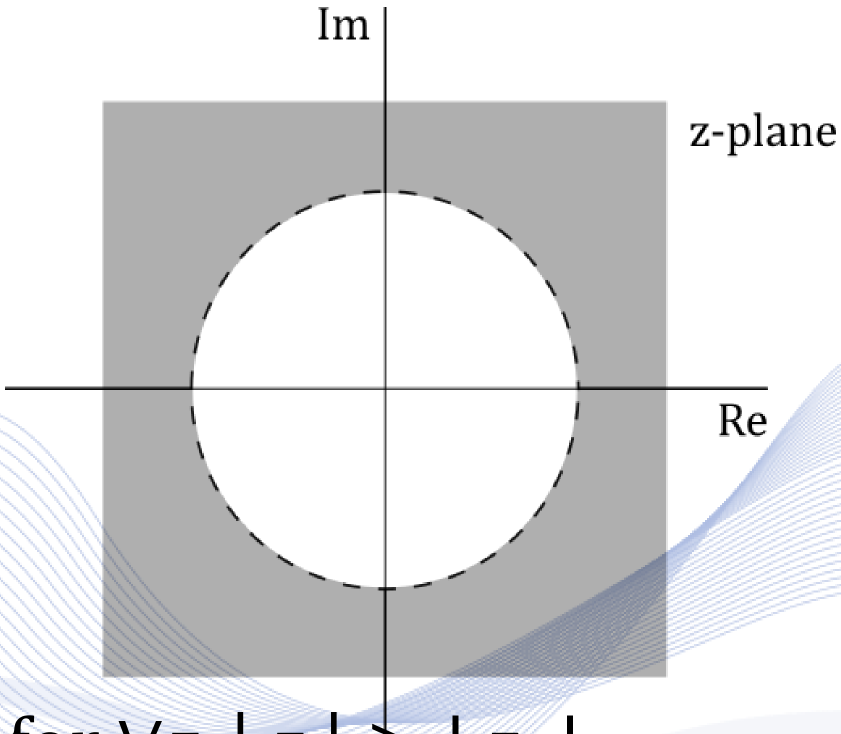
- As z is a complex number, it is a complex transform (mapping) of the form $\mathbb{R} \rightarrow \mathbb{C}$.
- If we consider z^{-1} as polynomial variable, then $X(z)$ can be considered to be a ***polynomial***.

Region of Convergence

- Right-hand sequence $x(n) = 0, n < n_1$:

Given a z_1 for which:

$$\sum_{n=n_1}^{\infty} |x(n) z_1^{-n}| < \infty$$



and given that $R_1 = |z_1|$. Then:

$$\sum_{n=n_1}^{\infty} |x(n) z^{-n}| < \sum_{n=n_1}^{\infty} |x(n) z_1^{-n}| < \infty, \qquad \text{for } \forall z, |z| > |z_1|$$

- ROC is the outside of a circle with radius R_1 except: $z = \infty$, if $n_1 < 0$.

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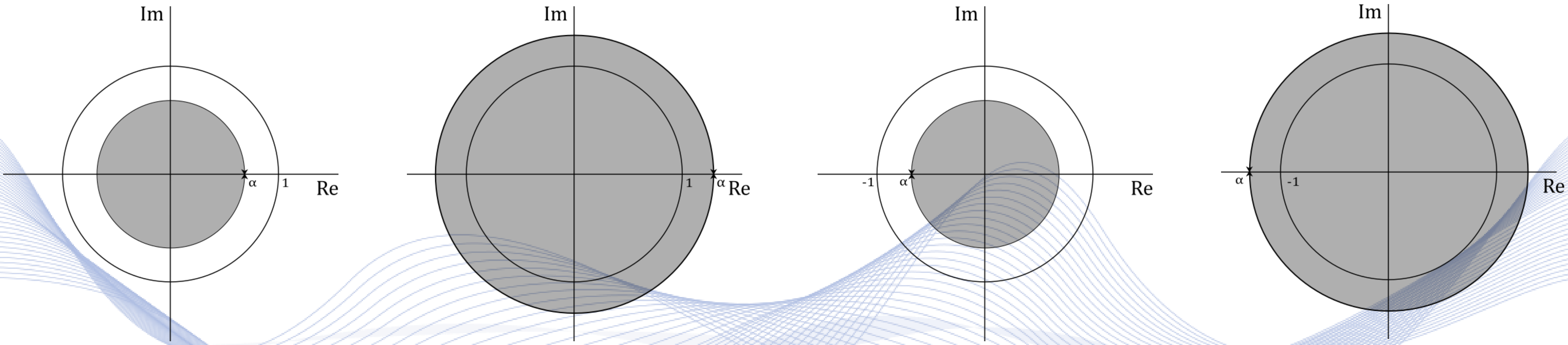
Region of Convergence

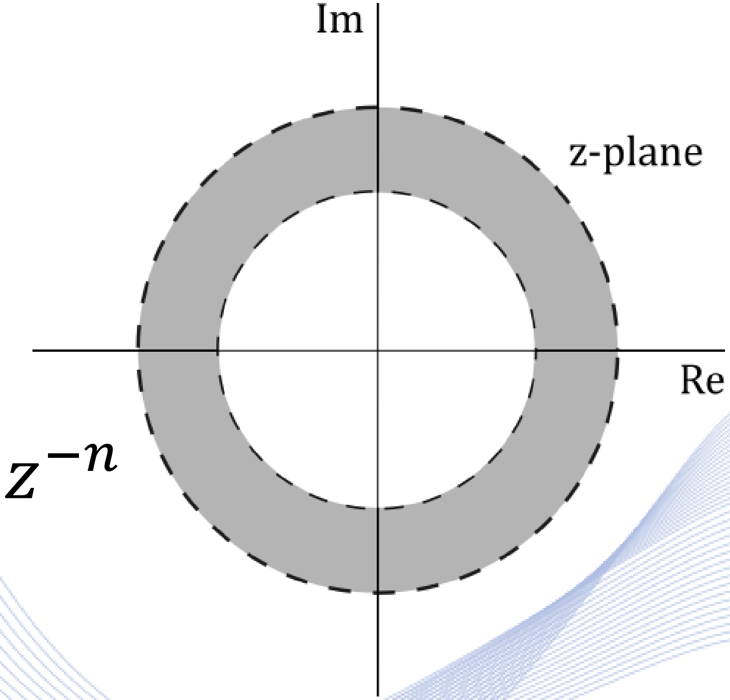


$0 < \alpha < 1$ | $\alpha > 1$ | $-1 < \alpha < 0$ | $\alpha < -1$

ROC of $X(z)$ for different values of α.

Region of Convergence

4. Bilateral sequence $(-\infty < n < \infty)$:



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} x(n) z^{-n} + \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

The first term converges for $R_1 < |z|$, the second one for $R_2 > |z|$.

ROC for $Y(z)$ is $R_1 < |z| < R_2$, if $R_1 < R_2$

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Inverse $\mathcal{Z}$ transform

Therefore, by combining the two relations, we can find:

$$\frac{1}{2\pi i}\oint_c X(z)z^{k-1}dz = x(k).$$

Inverse $\mathcal{Z}$ transform:

$$x(n) = \frac{1}{2\pi i}\oint_C X(z)z^{n_1}dz.$$

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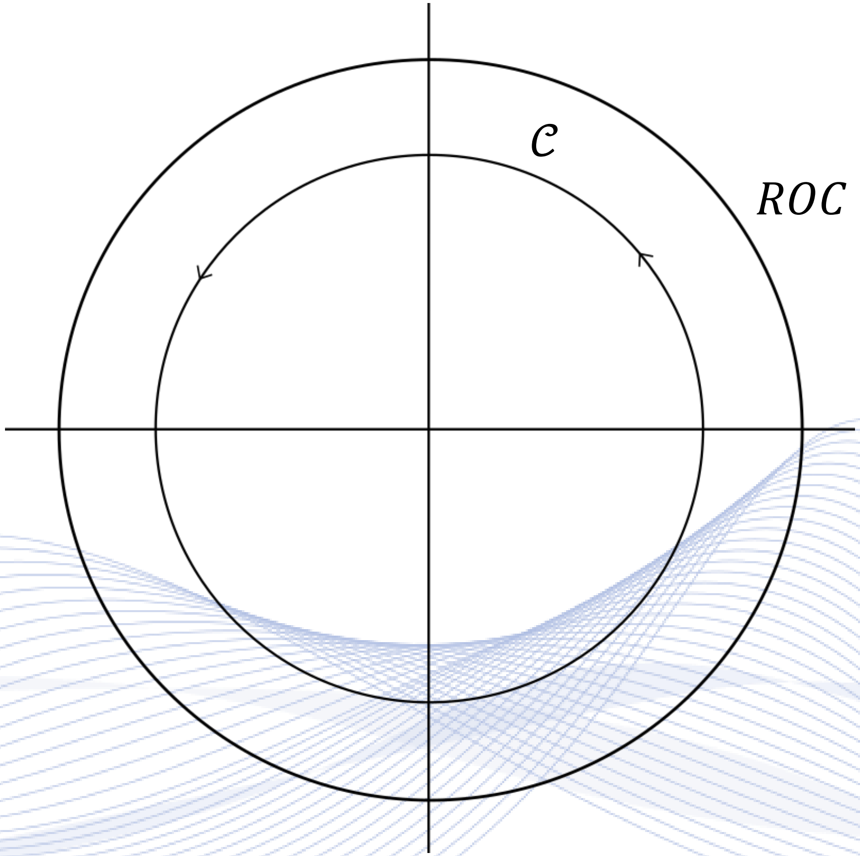
Inverse $\mathcal{Z}$ transform



Inverse $\mathcal{Z}$ transform integration.

$\mathcal{Z}$ transform

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Properties

Initial value theorem:

If $x(n) = 0$ for $n < 0$, then:

$$x(0) \leftrightarrow \lim_{z \to \infty} X(z).$$

Convolution:

$$y(n) = x(n) * h(n) \Rightarrow Y(z) = X(z)H(z).$$

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Transfer Function

An IIR system is described by a ***difference equation***:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k),$$

having coefficients $a_k, k = 0, \ldots, N$ and $b_k, k = 0, \ldots M$. By applying the time shifting property of $\mathcal{Z}$ transform, we get:

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k}.$$

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Transfer Function

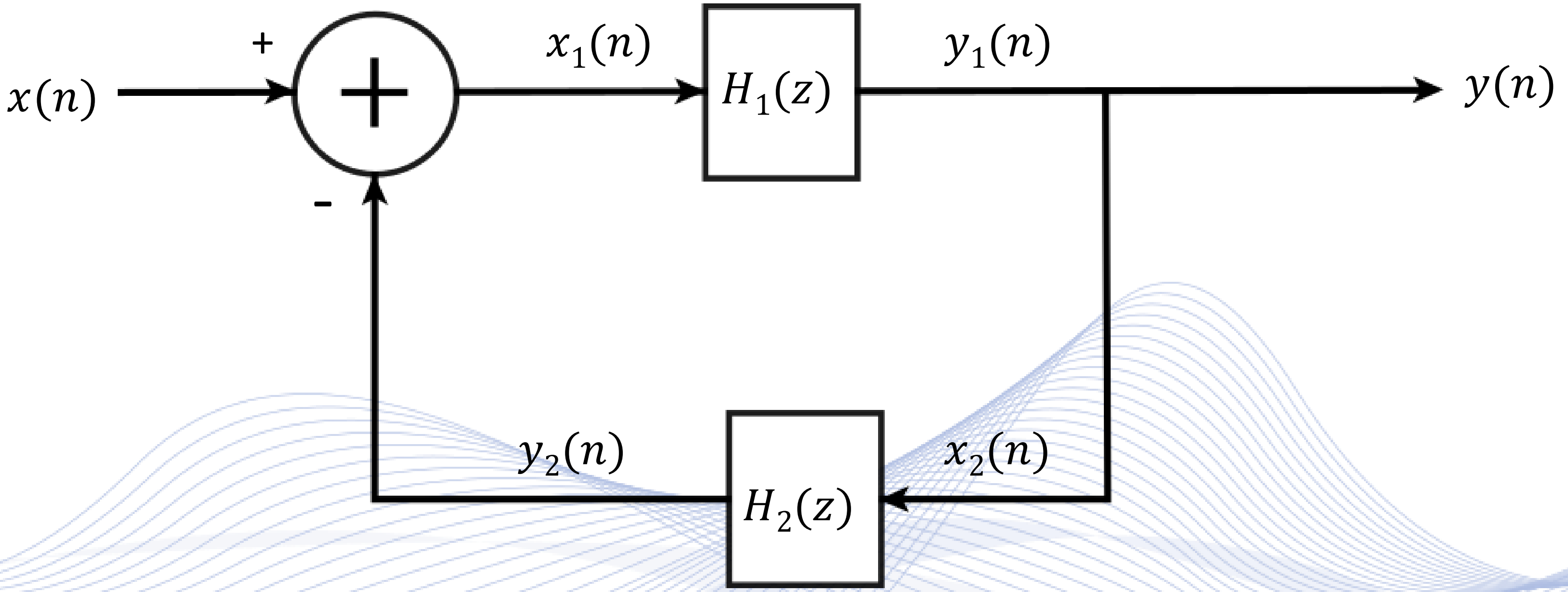
Therefore, IIR system ***transfer function*** is given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

System poles and zeros can be found by factorizing $H(z)$:

$$H(z) = H\frac{\prod_{k=0}^{M}(1 - c_k z^{-1})}{\prod_{k=0}^{N}(1 - d_k z^{-1})} = Hz^{N-M}\frac{\prod_{k=0}^{M}(z - c_k)}{\prod_{k=0}^{N}(z - d_k)}.$$

Transfer Function

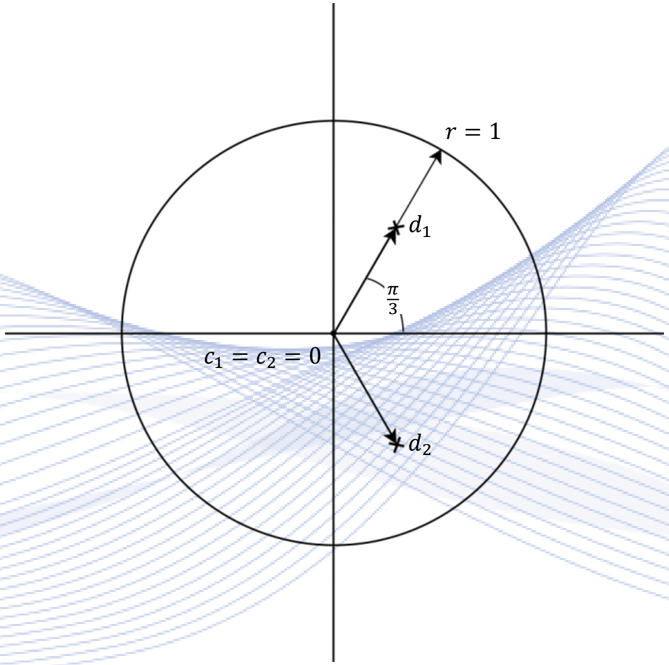


For two systems connected in a feedback loop, the total transfer function is:

$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}.$$

Transfer Function

- Since both poles $d_1 = 0.5e^{-i\frac{\pi}{3}}$ and $d_2 = 0.5e^{-i\frac{\pi}{3}}$ are within the unit circle: $|d_1| = |d_2| = 0.5 < 1$, the system is stable.



Transfer function zeroes and complex poles.

Transfer Function

- FIR systems have only zeroes, no poles (and no stability problems).

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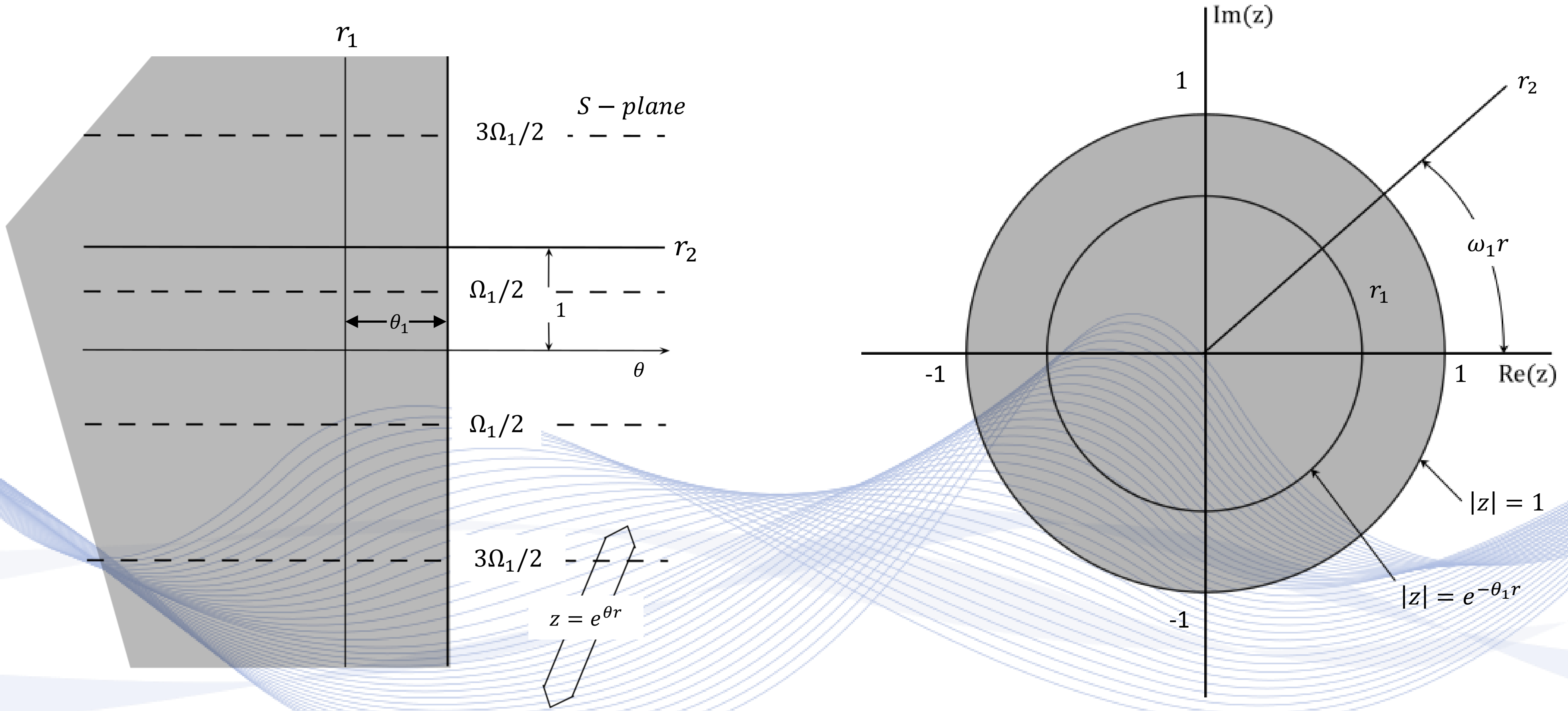
$\mathcal{Z}$ transform and Laplace Transform



Mapping plane s to plane z

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Q & A

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