Statistical Detection

A. Tsanakas, Prof. Ioannis Pitas
Aristotle University of Thessaloniki
pitas@csd.auth.gr
www.aiia.csd.auth.gr
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Artificial Intelligence &
Information Analysis Lab

Detection Theory

- **Introduction**
- Signal or Noise Decision
- Cost Function
- Likelihood Ratio Test
- MAP Detector
- Neyman-Pearson Hypothesis Testing

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Introduction

Signal detection theory is a means to measure the ability to differentiate between information-bearing patterns and random patterns that distract from the information.

Introduction

- ***Example***: A bit, 0 or 1, is sent through a noisy channel of a communication channel. The noise is modeled as a realization of a $N\,(0,1)$ random variable The receiver gets the bit plus noise. Assume that $P_0 = P_1 = \frac{1}{2}$. We must decide between two hypotheses, when the x received:

$$H_0\colon X \sim N(0,1)$$

$$H_1\colon X \sim N(1,1)$$

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Signal or Noise Decision

- A ***noisy signal*** x has the following representation:

$$x = s + n,$$

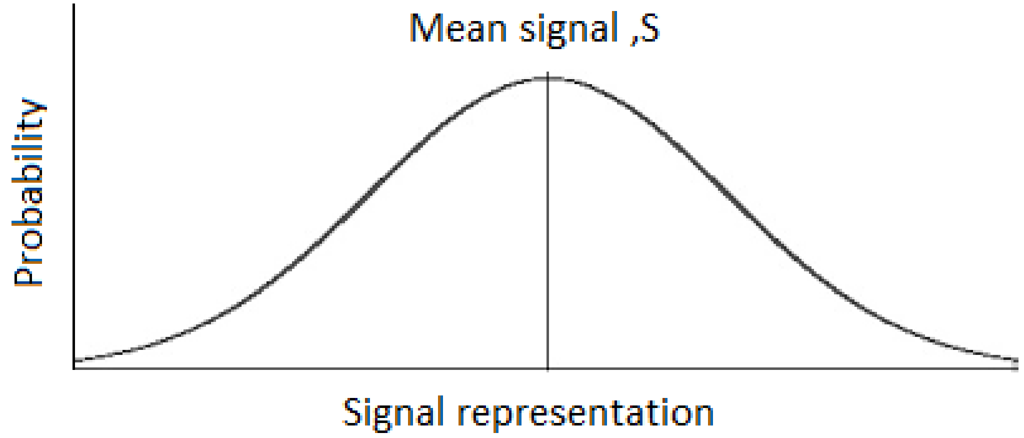
  - where s is the clean signal and n is the noise.

- The problem we want to deal with is how we can decode the signal s from x.

Signal or Noise Decision

- If noise is ***white***:

1. The received noisy signal will distribute as a normal distribution
2. The mean of this distribution would reflect the signal



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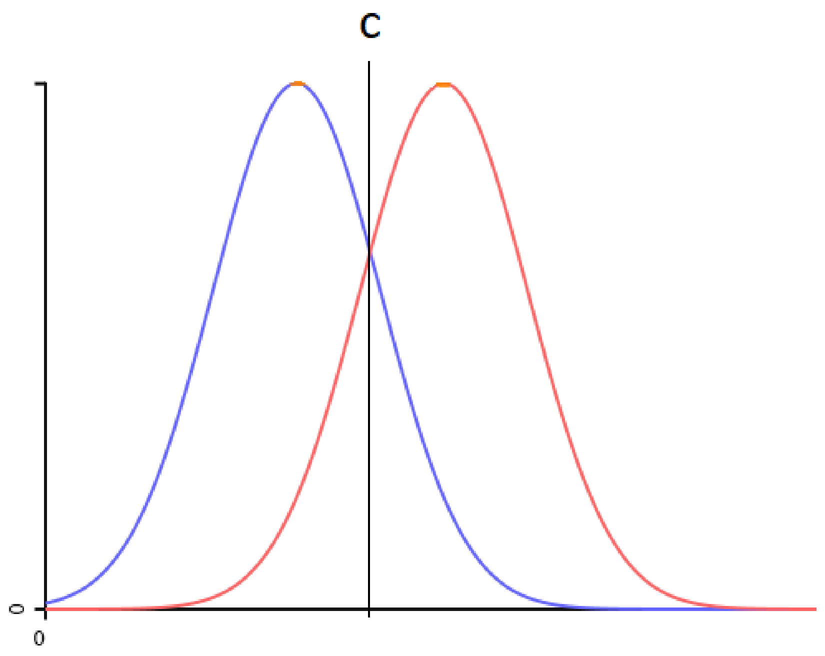
Signal or Noise Decision



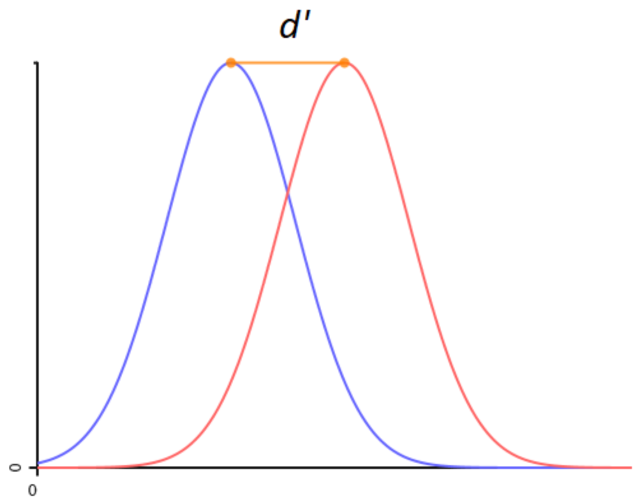
Decision can be made based on the information from two distribution. A simple criterion to make decisions is the following:

- If $x < C$: The received signal is noise.
- If $x > C$: The original signal is received.

Signal or Noise Decision

- ***Sensitivity*** (***Discriminability***) is a measure of how close are signal and noise. We can estimate the Sensitivity d' as the difference between means of s and n by the following type:

$$d' = \mu_s - \mu_n.$$



Signal or Noise Decision

		Stimuli	
		Signal	Noise
Response	Yes	Hit	False Alarm
	No	Miss	Correct Rejection

Different combinations of stimuli and responses.

Signal or Noise Decision

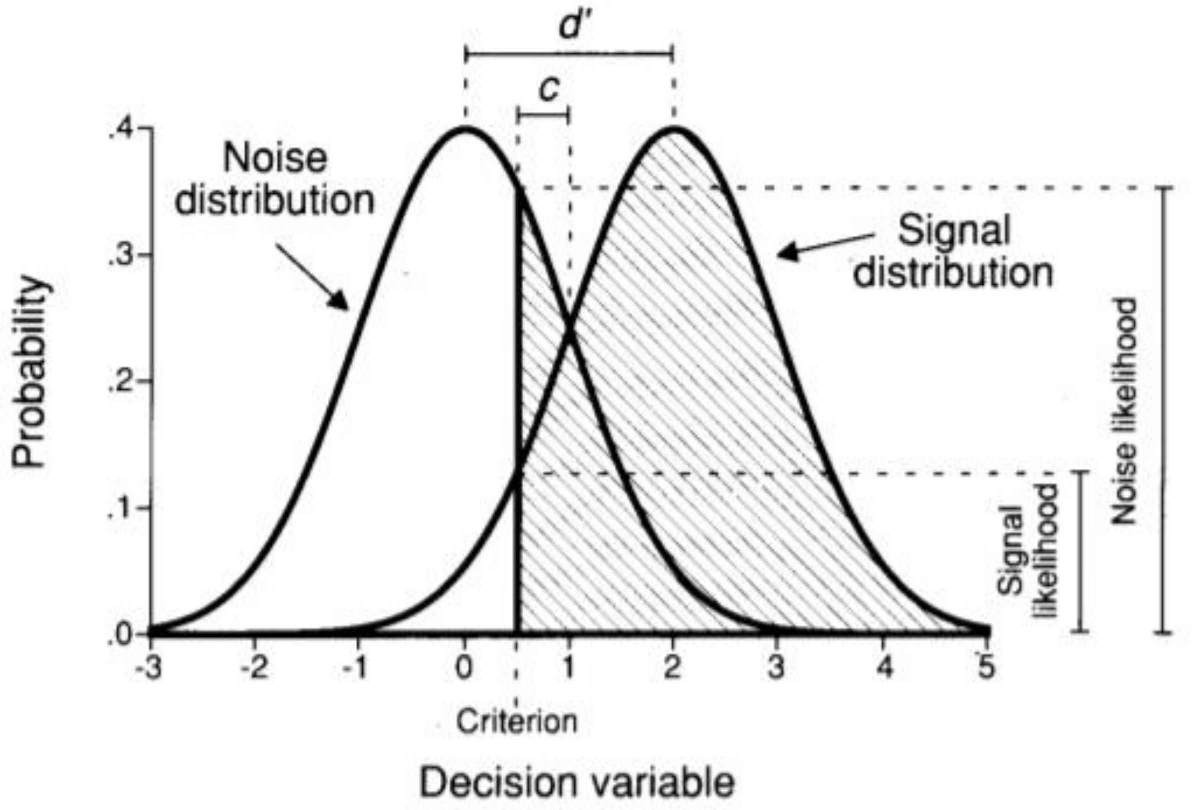


Figure: Distribution of the decision variable across noise and signal trials.

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Cost Function

- We want to design a decision rule $\hat{H}(x)$. For this purpose, we create two disjoint regions:

$$\mathcal{R}_0 = \{x : \hat{H}(x) = H_0\}$$

$$\mathcal{R}_1 = \{x : \hat{H}(x) = H_1\}$$

- To optimize the choice of decision regions, we can specify a cost for decisions.

Cost Function

- The ***expected Bayes Cost*** defined as:

$$C = \sum_{i,j=0}^{1} c_{ij} P(decided\ H_i,\ H_j\ is\ true)$$

$$= \sum_{i,j=0}^{1} c_{ij} \pi_j\ P(decided\ H_i | H_j\ is\ true).$$

where π_j is the probability the hypothesis H_j is true.

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Likelihood Ratio Test

- Therefore, the ***optimal test*** takes the following Likelihood Ratio form:
- For H_1:

$$L(x) = \frac{p_1(x)}{p_0(x)} > \frac{\pi_0 \, (c_{10} - c_{00})}{\pi_1 (c_{01} - c_{11})},$$

and for H_0:

$$L(x) = \frac{p_1(x)}{p_0(x)} < \frac{\pi_0 \, (c_{10} - c_{00})}{\pi_1 (c_{01} - c_{11})}.$$

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MAP Detector

Finally, the MAP detector has the following form:

- For H_1:

$$\mathrm{P}(H = H_1|x) > P(H = H_0|x),$$

and for H_0 :

$$\mathrm{P}(H = H_1|x) < P(H = H_0|x).$$

- This is also called ***Maximum a Posteriori*** (MAP) Detector.

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Neyman-Pearson Hypothesis Testing

We defined the following probabilities:

- ***Hit*** probability: $P(\widehat{H} = H_1 | H_1)$.
- ***Correct Rejection*** probability: $P(\widehat{H} = H_0 | H_0)$.
- ***False Alarm*** probability: $P_{FA} = P(\widehat{H} = H_1 | H_0) = \int_{\mathcal{R}_1} p_0(x)dx$.
- ***Mis-detection*** probability: $P_{MD} = P(\widehat{H} = H_0 | H_1) = \int_{\mathcal{R}_0} p_1(x)dx$.

$$P_{MD} = 1 - P_D.$$

Neyman-Pearson Hypothesis Testing

The ***Neyman-Pearson*** criterion is defined as follows.

- Minimize the probability $P_{MD} = P(\hat{H} = H_0 | H_1)$ subject to:

$$P_{FA} < a.$$

- The main advantage of this minimization criterion is that it does not require prior probabilities nor cost assignments.

Neyman-Pearson Hypothesis Testing

To ***minimize*** the probability P_{MD}, we rely on the following *theorem*:

To ***maximize*** P_D *with a given* $P_{FA} < a$ *, decide* H_1 *if*

$$L \doteq \frac{p(x|H_1)}{p(x|H_0)} \geq \lambda$$

where λ found from

$$P_{FA} = \int_{x:L(x)>\lambda} p(x|H_0)dx = a$$

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Q & A

Thank you very much for your attention!

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Contact: Prof. I. Pitas
pitas@csd.auth.gr