# State –Space Equations summary

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# State – Space Equations

- **Multiple Input-Output Systems**
- Single Input-Output Systems
- RNNs

# Definition

**State – Space equations** are a generalization of difference equations in digital filters. Through them, we can better understand the dynamic behavior of a filter. Additionally, they lead to important implementation structures of digital filters.

# Multiple Input-Output systems

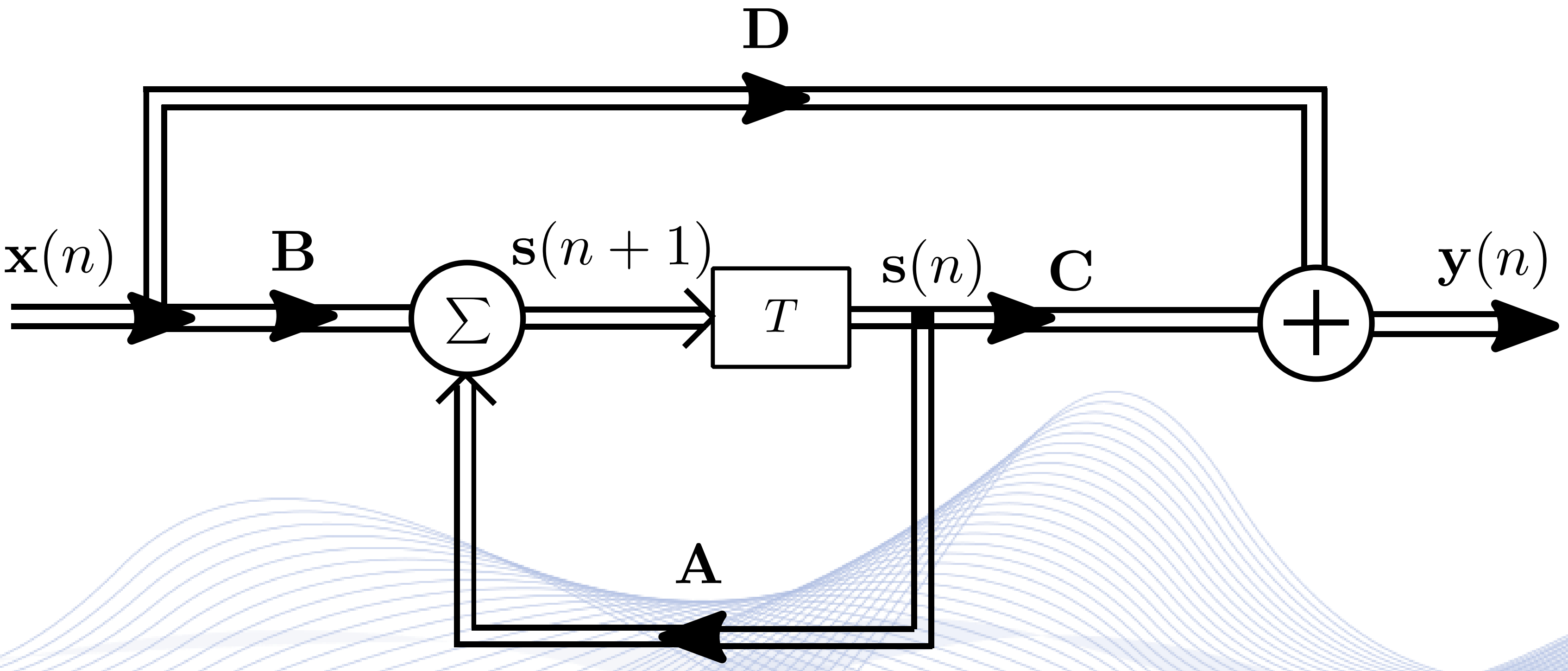
***State-space equations*** of a time-invariant ***Multiple Input-Multiple Output*** (***MIMO***) system:

$$\mathbf{s}(n+1) = \mathbf{A}\mathbf{s}(n) + \mathbf{B}\mathbf{x}(n),$$

$$\mathbf{y}(n) = \mathbf{C}\mathbf{s}(n) + \mathbf{D}\mathbf{x}(n).$$

- $\mathbf{s}(n) = [s_1(n), \ldots.., s_r(n)]^T \in \mathbb{R}^r$: state vector having $r$ states.
- $\mathbf{x}(n) \in \mathbb{R}^p$: $p$ – channel input signal vector
- $\mathbf{y}(n) \in \mathbb{R}^q$: $q$ – channel output signal vector.
- Matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ have $r \times r, r \times p, q \times r, q \times p$ dimensions, respectively.

# Multiple Input-Output systems




Multiple Input-Output system.

# State – Space Equations

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- **Single Input-Output Systems**
- RNNs

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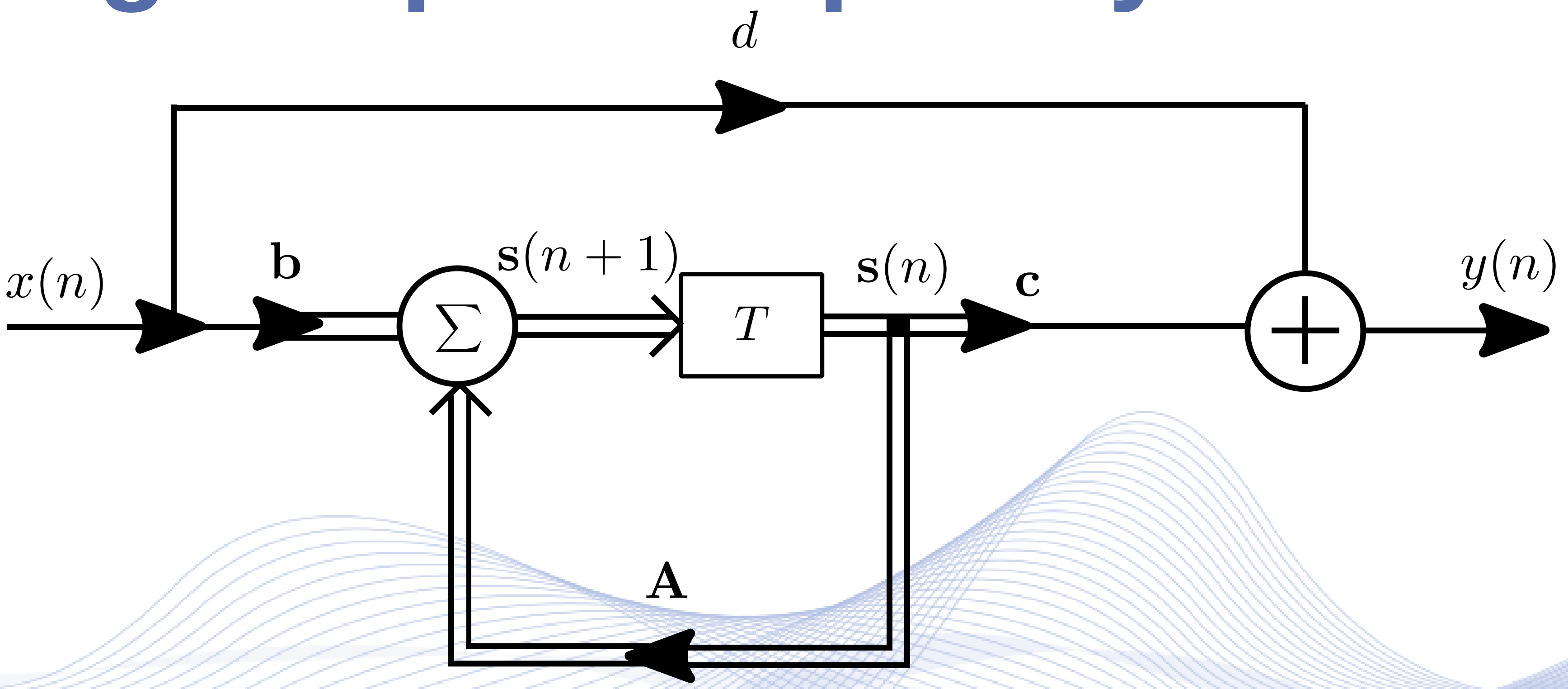
# Single Input-Output Systems

A ***Single Input-Output*** IIR system can be described using the linear equations:

$$\mathbf{s}(n+1) = \mathbf{A}\mathbf{s}(n) + \mathbf{b}x(n),$$
$$y(n) = \mathbf{c}^T\mathbf{s}(n) + dx(n).$$

- $\mathbf{s}(n) = [s_1(n), \dots, s_r(n)]^T \in \mathbb{R}^r$: state vector having $r$ states.
- $x(n) \in \mathbb{R}$: single-channel input signal.
- $y(n) \in \mathbb{R}$: single-channel output signal vector.
- Matrix $\mathbf{A}$ has $r \times r$ dimensions.
- Vectors $\mathbf{b}$, $\mathbf{c}$ have dimensions $r$.
- $d$ is a scalar parameter.

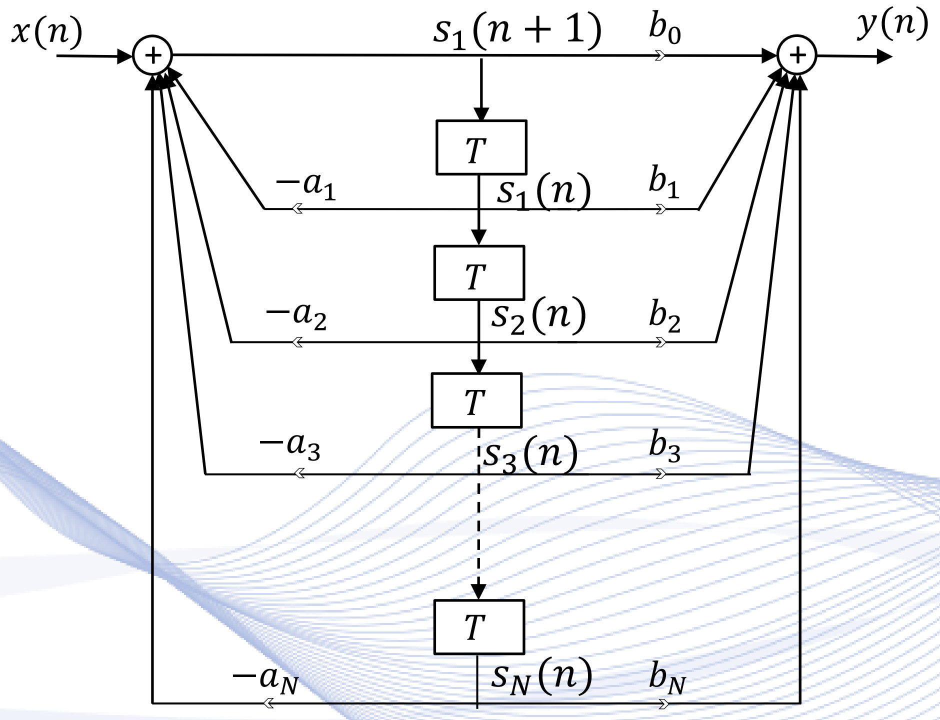
# Single Input-Output Systems




Single Input-Output system.

# Single Input-Output Systems

### *Direct Filter Structure II*




- The delay lines are merged into one.
- Minimum number of delays is achieved.
- The output of each delay is a state $s_i(n), i = 1, \dots, N+1$.

# Single Input-Output Systems

***State-space equations of a general IIR filter*.**

State vector $\mathbf{s}^T(n) = [s_1(n), \dots, s_N(n)]$.

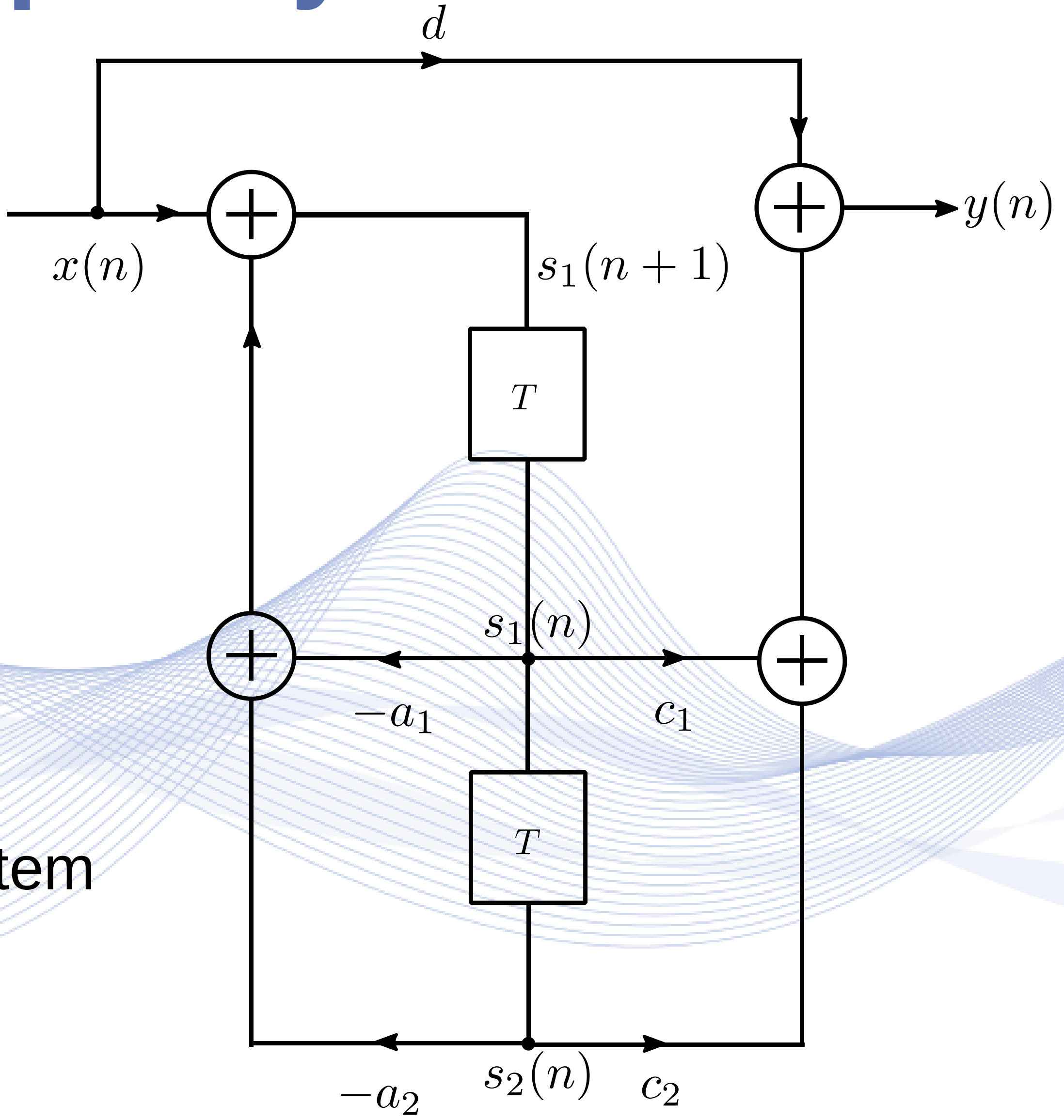
$$s_1(n+1) = -\sum_{k=1}^{N} a_k s_k(n) + x(n)$$

$$s_2(n+1) = s_1(n), \qquad \dots, \qquad s_N(n+1) = s_{N-1}(n)$$

$$y(n) = b_0 x(n) + \sum_{k=1}^{N} b_k s_k(n) =$$

$$b_0 x(n) + \sum_{k=1}^{N} (b_k s_k(n) - b_0 a_k s_k(n))$$

# Single Input-Output Systems




A second-order IIR system and its state variables.

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# Single Input-Output Systems

***Example***

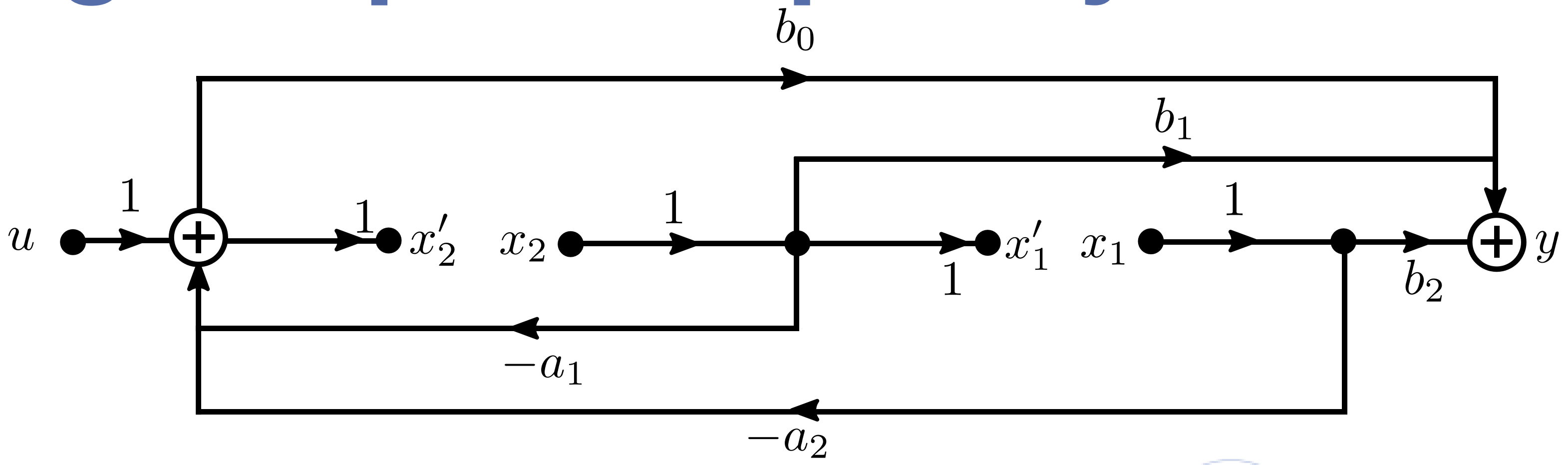
- A second order ***Infinite Impulse Response*** (*IIR*) system can be described by its ***states*** $s_1(n), s_2(n)$ are the outputs of the delay units.
- They are given by:

$$s_1(n+1) = -a_1 s_1(n) - a_2 s_2(n) + x(n),$$
$$s_2(n+1) = s_1(n).$$

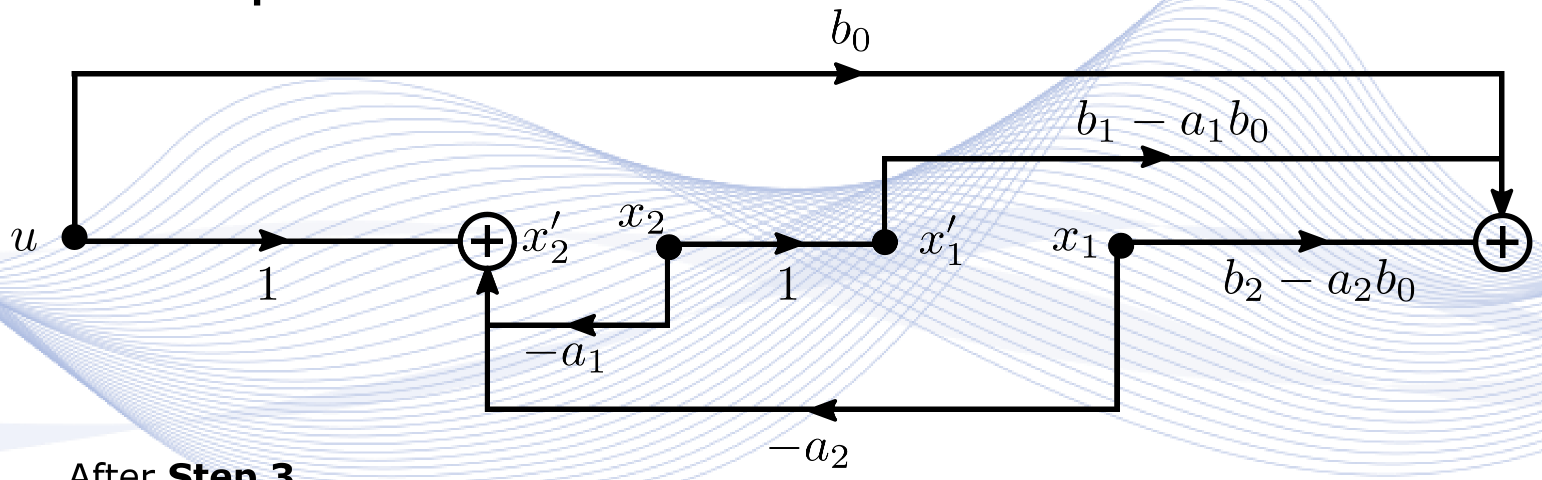
- The system output is:

$$y(n) = c_1 s_1(n) + c_2 s_2(n) + dx(n).$$

# Single Input-Output Systems



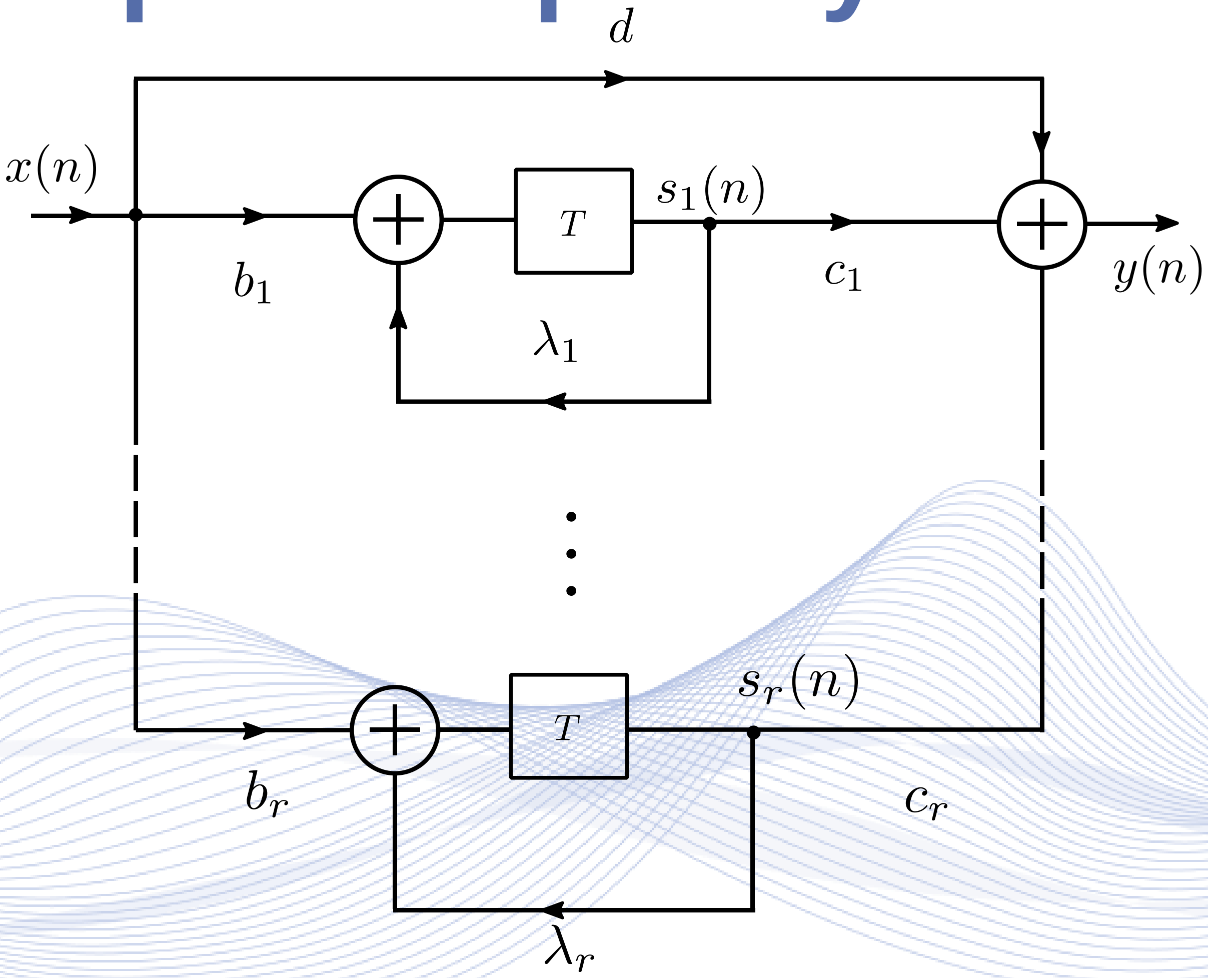

After **Step 2.**




After **Step 3.**

Example of a 2D IIR system.

# Single Input-Output Systems




IIR state-space system implementation.

# State – Space Equations

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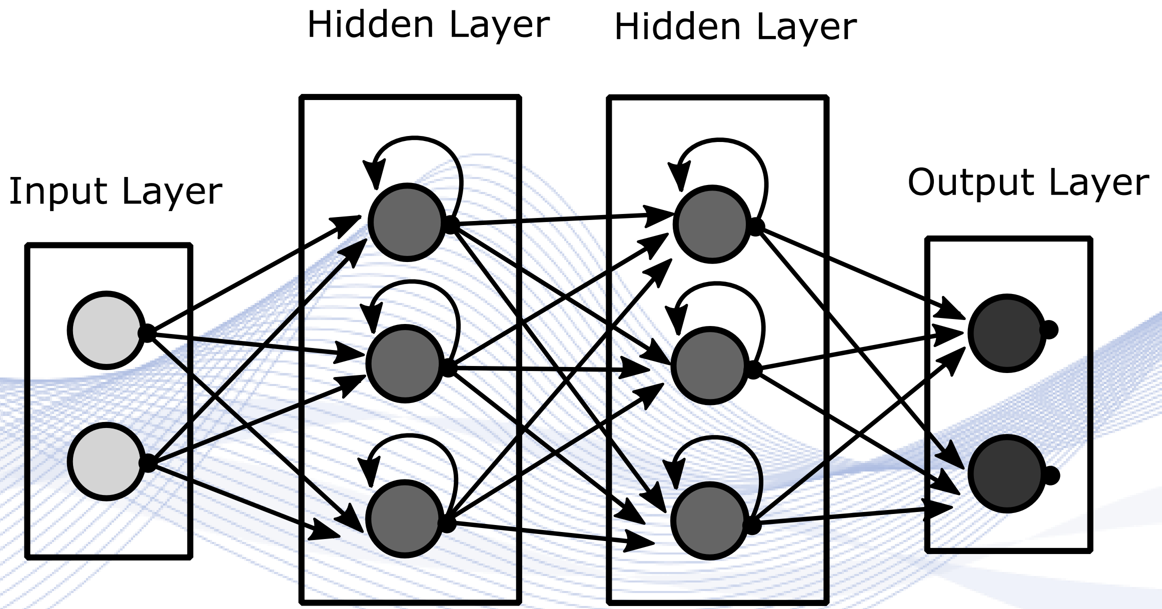
- **RNNs**

# RNNs

- **RNN is a type of neural network composed of:**
- Input neurons.
- Hidden neurons.
- Output neurons.




RNN structure

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# RNNs

***RNN neurons***

Input : $\mathbf{x}(n) \in \mathbb{R}^p$ and $\mathbf{W}_{xs} \in \mathbb{R}^{p \times r}$

Output: $\mathbf{y}(n) \in \mathbb{R}^q$ and $\mathbf{W}_{sy} \in \mathbb{R}^{r \times q}$

Hidden state: $\mathbf{s}(n-1) \in \mathbb{R}^r$ and $\mathbf{W}_{ss} \in \mathbb{R}^{r \times r}$

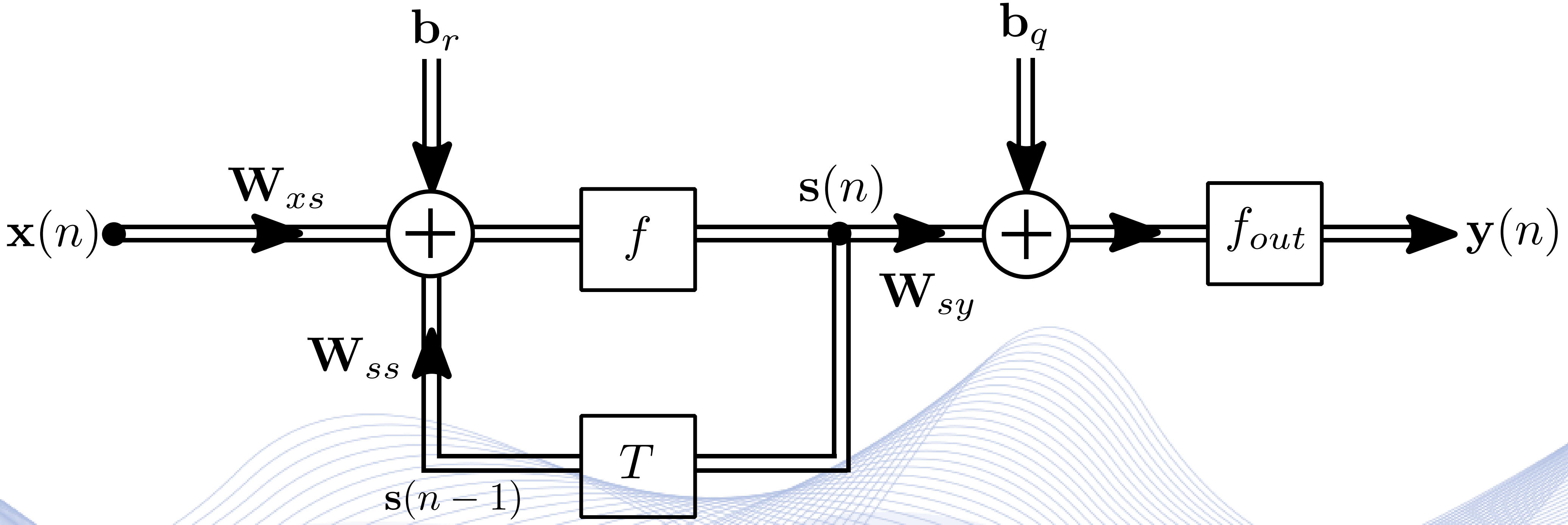
Biases : $\mathbf{b}_r \in \mathbb{R}^r$ and $\mathbf{b}_q \in \mathbb{R}^q$

$$\mathbf{s}(n) = f(\mathbf{W}_{xs}\,\mathbf{x}(n) + \mathbf{W}_{ss}\mathbf{s}(n-1) + \mathbf{b}_r)$$

$$\mathbf{y}(n) = f_{out}(\mathbf{W}_{sy}\mathbf{s}(n) + \mathbf{b}_q)$$

- $f_{out}$: Activation function of output layer.
- $f$: Hidden state’s activation function; non-linearity.

# RNNs




RNN neuron architecture.

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# Q & A

Thank you very much for your attention!

More material in
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