Spectral Signal Analysis summary

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Spectral Signal Analysis

Spectral Signal Analysis is based on the content of the signal frequency.

Basic tools for the analysis:

- Fourier Transform
- Power Spectrum

Spectral Signal Analysis

- **Power Spectrum**
- Power Spectrum: Random Signals
- Bartlett Method
- Welch Method
- Blackman – Tukey Method

Power Spectrum

Suppose that a deterministic analog signal $x_a(t)$ is sampled with frequency sampling f_s.

The energy E of the analog signal with finite power is:

$$E = \int_{-\infty}^{\infty} |x_a(t)|^2 dt < \infty$$

Power Spectrum

The Fourier transform is given by:

$$X_a(F) = \int_{-\infty}^{\infty} x_a(t) e^{-j2\pi Ft} dt$$

Based on the Parseval theorem:

$$E = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} |X_a(F)|^2 dF$$

Power Spectrum

The Power Spectrum is equal to:

$$S_{xx}(F) = |X_a(F)|^2$$

The auto – correlation function $R_{xx}(\tau)$ of the signal $x_a(t)$ is:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x_a^*(t)x_a(t+\tau)dt$$

Power Spectrum

The Power Spectrum is equal to the Fourier transform of the function:

$$S_{xx}(F) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j2\pi \mathrm{F}\tau} d\tau$$

Power Spectrum

Indirect Calculation Method

The auto – correlation function $r_{xx}(k)$ is transformed resulting to the calculation of the power spectrum $S_{xx}(f)$.

There are N samples of signal $x(n),\ 0 \leq n \leq N-1$.

This is equal to the multiplication:

$$\tilde{x} = x(n)w(n) = \begin{cases} x(n), & 0 \leq n \leq N-1 \\ 0, & otherwise \end{cases}$$

Power Spectrum

The power spectrum is calculated with the use of $\boldsymbol{DFT}$ and $\boldsymbol{FFT}$:

$$S_{\tilde{X}\tilde{X}}(f)|_{f=k/N} = S_{\tilde{X}\tilde{X}}\left(\frac{k}{N}\right) = |\tilde{X}(k)|^2 = |\sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j2\pi kn/N}|^2$$

Spectral Signal Analysis

- Power Spectrum
- **Power Spectrum: Random Signals**
- Bartlett Method
- Welch Method
- Blackman – Tukey Method

Power Spectrum: Random Signals

Stationary random signals don't have finite energy.

That's why they can only be characterized by the power density spectrum.

The power spectrum is defined by the auto – correlation function $\gamma_{xx}(\tau)$ of the signal $x(t)$:

$$\gamma_{xx}(\tau) = E[x^*(t)x(t+\tau)]$$

Power Spectrum: Random Signals

Based on ***Wiener – Khintchine*** theorem, the power spectrum is the Fourier transform of the auto – correlation function $\gamma_{xx}(\tau)$:

$$\Gamma_{xx}(F) = \int_{-\infty}^{\infty} \gamma_{xx}(t) e^{-j2\pi Ft} dt$$

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Power Spectrum: Random Signals

If the observation interval is $[-T_0, T_0]$, an estimating of the auto – correlation function is:

$$R_{xx}(\tau) = \frac{1}{2T_0}\int_{-T_0}^{T_0} x^*(t)x(t+\tau)dt$$

Power Spectrum: Random Signals

A ***periodogram*** is the estimating function of the power spectrum.

The estimated value of the periodogram is:

$$E[P_{xx}(f)] = E\left[\sum_{m=-N+1}^{N-1} r_{xx}(m)\, e^{-j2\pi f m}\right] =$$

$$= \sum_{m=-N+1}^{N-1} E[r_{xx}(m)]\, e^{-j2\pi f m} =$$

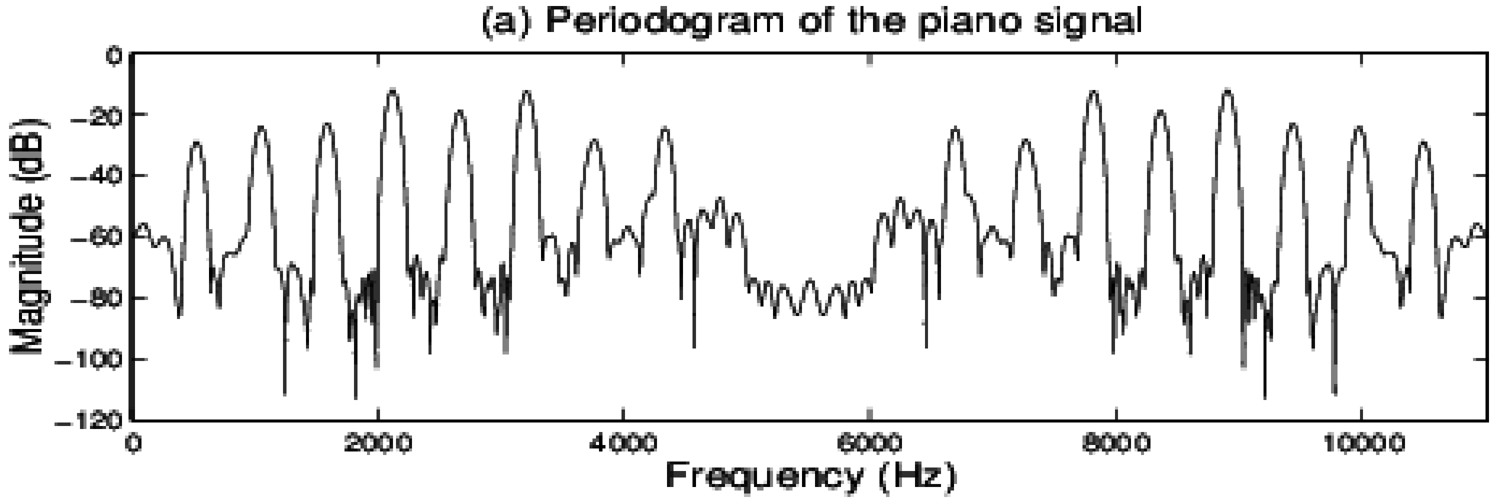
$$= \sum_{m=-N+1}^{N-1} \left(1 - \frac{|m|}{N}\right) \gamma_{xx}(m)\, e^{-j2\pi f m}$$

Power Spectrum: Random Signals

The periodogram can be calculated using $\boldsymbol{DFT}$:

$$P_{xx}\left(\frac{k}{N}\right) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}\right|^2, k = 0, 1, \ldots, N-1$$

Power Spectrum: Random Signals



A periodogram of the piano signal [BAD2006].

Spectral Signal Analysis

- Power Spectrum
- Power Spectrum: Random Signals
- **Bartlett Method**
- Welch Method
- Blackman – Tukey Method

Power Spectrum: Bartlett

The high variability of the periodogram can be decreased if the sequency $x(n)$ is broken down to K subsequences, of M length without overlapping ($N = KM$):

$$x_i(n) = x(n + iM), \qquad i = 0, \dots, K-1, \qquad n = 0, \dots, M-1$$

The periodogram of each subsequence is:

$$P_{xx}^{(i)}(f) = \frac{1}{M} \left| \sum_{n=0}^{M-1} x_i(n)\, e^{-j2\pi f n} \right|^2, \qquad i = 0, \dots, K-1$$

Power Spectrum: Bartlett

The new estimating function is equal to the average of the periodograms:

$$P_{xx}^{B}(f) = \frac{1}{K}\sum_{i=0}^{K-1} P_{xx}^{(i)}(f)$$

The average value of the estimating function is:

$$E[P_{xx}^{B}(f)] = \frac{1}{K}\sum_{i=0}^{K-1} E[P_{xx}^{(i)}(f)] = E[P_{xx}^{(i)}(f)]$$

Spectral Signal Analysis

- Power Spectrum
- Power Spectrum: Random Signals
- Bartlett Method
- **Welch Method**
- Blackman – Tukey Method

Power Spectrum: Welch

Welch method uses L overlapping subsequences:

$$x_i(n) = x(n + iD), \qquad n = 0, \dots, M-1, \qquad i = 0, \dots, L-1$$

If $D = M$, the subsequences do not overlap/

If $\mathrm{D} = M/2$, there is 50% overlapping.

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Power Spectrum: Welch

In every subsequence a modified periodogram is used:

$$\tilde{P}_{xx}^{(i)}(f) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} x_i(n)w(n)e^{-j2\pi fn}\right|^2, \qquad i = 0, \dots, L-1$$

where

$$U = \frac{1}{M}\sum_{n=0}^{M-1} w^2(n)$$

Spectral Signal Analysis

- Power Spectrum
- Power Spectrum: Random Signals
- Bartlett Method
- Welch Method
- **Blackman – Tukey Method**

Power Spectrum: Blackman – Tukey

Method:

1. Calculation of estimating function
2. Multiplication with window $w(m)$
3. Fourier Transform

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Q & A

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