Orthogonal Signal Transforms. Fourier Series summary

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Version 3.5.1

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Orthogonal Signal Transforms

- **Vector calculus**
- Orthogonal functions
- Trigonometric Fourier Series
- Exponential Fourier Series
- Triangular Fourier Series
- Discrete Orthogonal Signals
- Discrete Fourier Transform
- Discrete Cosine Transform

Vector calculus

A signal $x(t)$, or $x(n), n = 0, \ldots, N-1$ can be represented by vectors. This is very easy for discrete-time signals:

$$\mathbf{x} = [x(0), \ldots, x(N-1)]^T \in \mathbb{R}^N.$$

There are major benefits for vectorial signal representations:

- Signal analysis is linked to geometry, e.g.,
  - signal similarity between two signals $\mathbf{x}, \mathbf{y}$ can be inversely proportional to Euclidean vector distance $||\mathbf{x} - \mathbf{y}||$.

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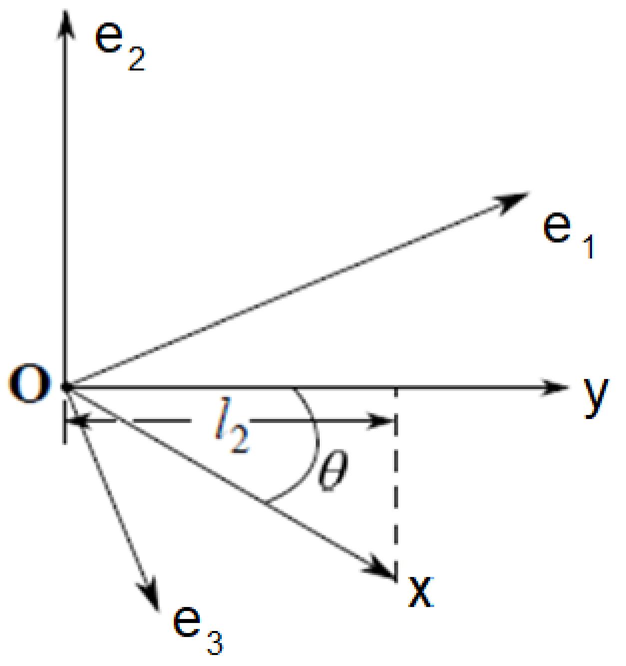
Vector calculus

Inner vector product or ***dot product*** in $\mathbb{R}^N$:

$\mathbf{x}^T\mathbf{y} = \mathbf{x}\mathbf{y}^T = \sum_{n=1}^{N} x_n\, y_n = ||\mathbf{x}|| \cdot ||\mathbf{y}|| \cos\theta$.

- θ: the angle formed by the two vectors.



Vector calculus

Properties of the inner vector product:

- It is a scalar value.
- Its value is equal to the product of the length of one vector $||\mathbf{x}||$ and the length of the projection of the second vector $l = ||\mathbf{y}|| \cos\theta$ on the first one.
- It is maximal for co-linear vectors of the same direction, i.e., for $\theta = 0$:

$$\mathbf{x}^T\mathbf{y} = ||\mathbf{x}|| \cdot ||\mathbf{y}||.$$

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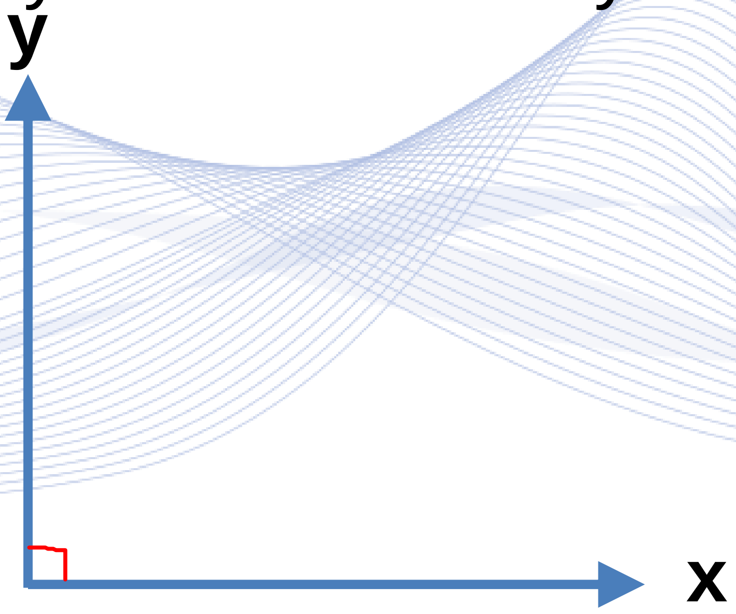
Orthogonal functions

- Two vectors $\mathbf{x}, \mathbf{y}$ are ***orthogonal***, if their inner product equals 0:

$$\mathbf{x}^T\mathbf{y} = \mathbf{x}\mathbf{y}^T = ||\mathbf{x}|| \cdot ||\mathbf{y}|| \cos\theta = 0,$$

or they are co-linear $\theta = 0$.

- The term orthogonality is used as a synonym for the term perpendicular.



Orthogonal functions

A vector $\mathbf{x}$ can be represented by its components $x_n, n = 1, \ldots, N$ in the Euclidean space $\mathbb{R}^N$:

$$\mathbf{x} = \sum_{n=1}^{N} x_n \, \mathbf{e}_n$$

Unit vectors $\mathbf{e}_n$, $n = 1, \ldots, N$ are ***perpendicular*** to each other and orthonormal:

$$\mathbf{e}_n^T \mathbf{e}_m = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

thus forming an ***orthonormal coordinate system*** in $\mathbb{R}^N$.

Orthogonal functions

Examples of orthogonal function systems:

- ***Trigonometric functions***:

$$\sqrt{\frac{1}{T}},\ \sqrt{\frac{2}{T}}\cos(n\Omega t),\ \sqrt{\frac{2}{T}\sin(n\Omega t)},\qquad \Omega=\frac{2\pi}{T},\qquad n=1,\ldots$$

Indeed, orthonormality holds:

$$\int_0^T \sqrt{\frac{2}{T}}\cos(n\Omega t)\,\sqrt{\frac{2}{T}}\cos(m\Omega t)\,dt=\begin{cases}1, & n=m\\ 0, & n\neq m\end{cases}.$$

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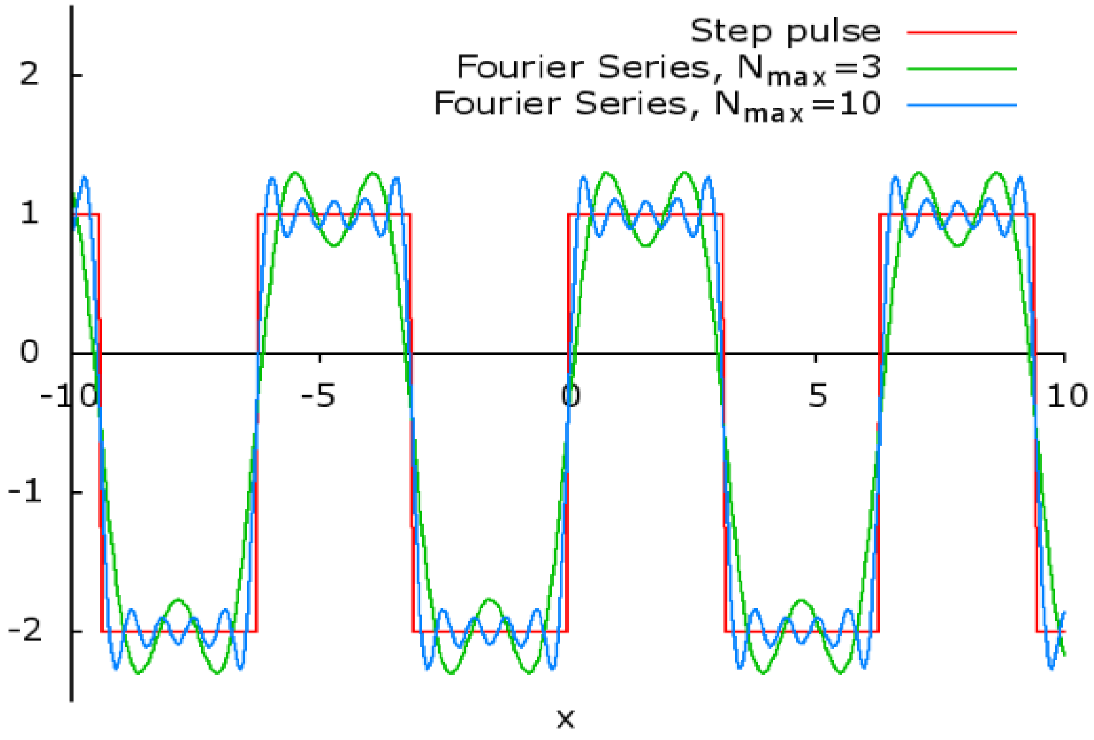
Trigonometric Fourier Series

Every periodic signal $x(t)$, with period T can be represented by a series of sines and cosines which are harmonically related.



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Trigonometric Fourier Series

- Signal $x(t)$ with period T can be represented as:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\Omega t + b_k \sin k\Omega t), \qquad \Omega = \frac{2\pi}{T}$$

- This forms the basis for ***harmonic signal analysis***.
- Frequencies $k\Omega = k\frac{2\pi}{T}$ are called ***harmonics***.
- a_k and b_k are the so-called ***Fourier coefficients***.
- We can calculate a_k and b_k as follows:

$$a_0 = \frac{1}{T}\int_T x(t)dt, \qquad a_k = \frac{2}{T}\int_T x(t)\cos k\Omega t\, dt, \qquad k = 1, \ldots$$

$$b_k = \frac{2}{T}\int_T x(t)\sin k\Omega t\, dt, \qquad k = 1, \ldots$$

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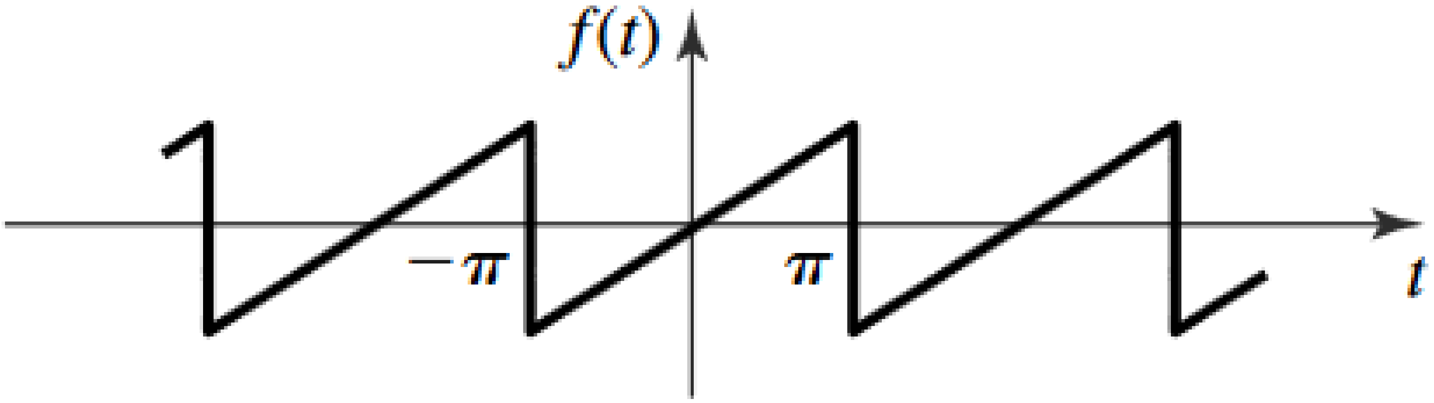
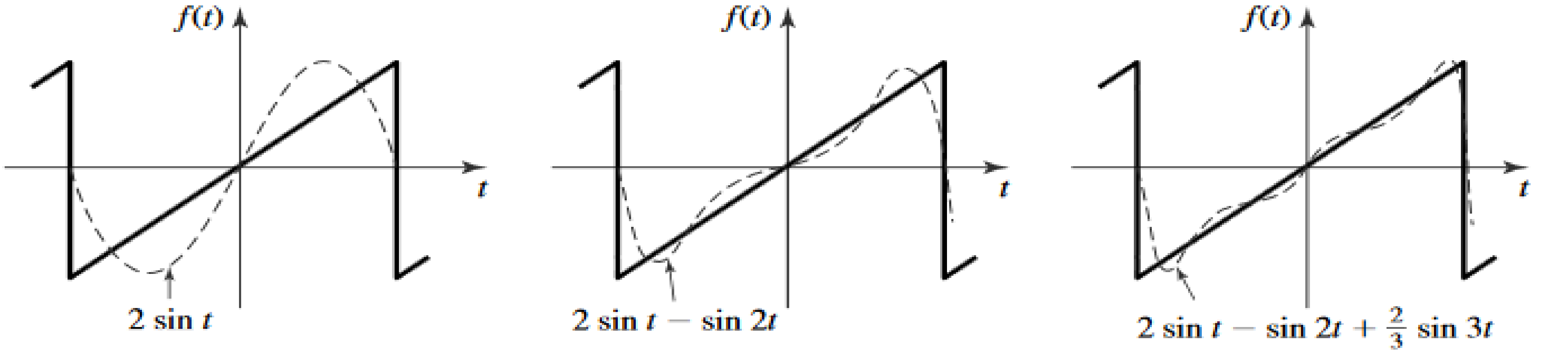
Trigonometric Fourier Series

Example: Find the Fourier Series representation of the periodic signal $x(t) = t, -\pi < t < \pi$ in one period.



Trigonometric Fourier Series



Fourier Series representation using increasing number of Fourier coefficients.

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Exponential Fourier Series

Every periodic signal $x(t)$, with period T, can be represented as a sum of complex exponentials:

$$x(t) = \sum_{-\infty}^{\infty} X_k e^{ik\Omega t}, \qquad \Omega = \frac{2\pi}{T},$$

- X_k are the so-called ***Fourier coefficients.***

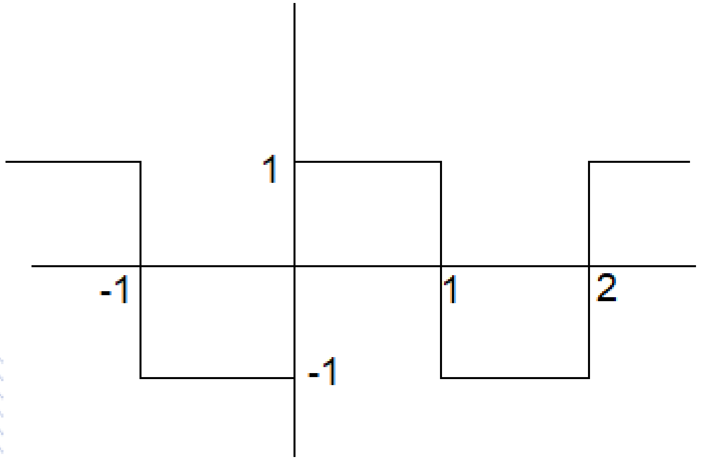
Relation between trigonometric and exponential Fourier Series:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\Omega t + b_k \sin k\Omega t) = \sum_{-\infty}^{\infty} c_k e^{ik\Omega t}.$$

Exponential Fourier Series

Example: Find the exponential series for the following rectangular wave with period $T = 2$:

$$x(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$$



Solution:

$$\Omega = \frac{2\pi}{T} = \pi.$$

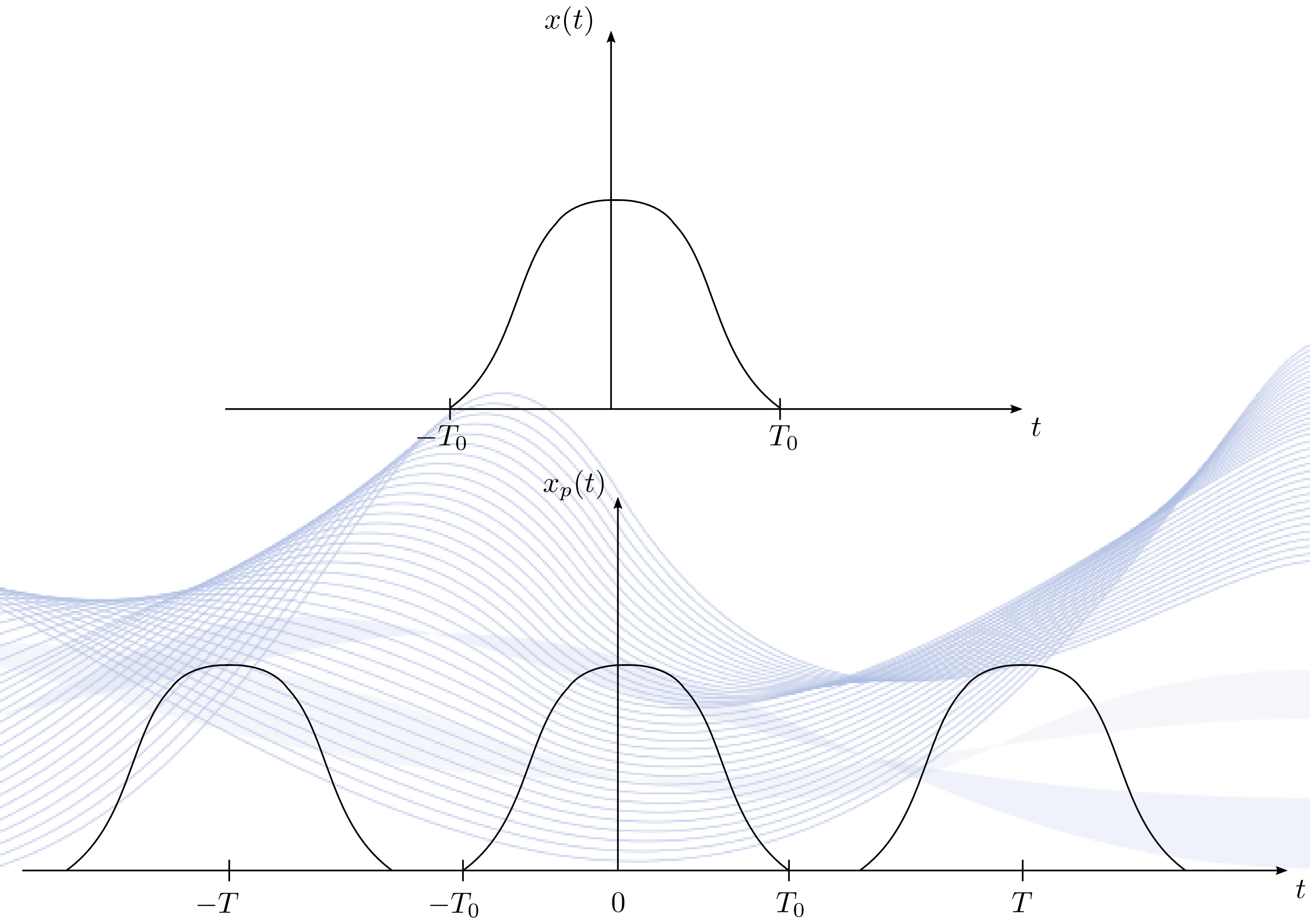
$$X_0 = \frac{1}{2}\int_{-1}^{1} x(t)dt = \frac{1}{2}\int_{-1}^{0} dt - \frac{1}{2}\int_{0}^{1} dt = 0.$$

Exponential Fourier Series

finite duration $[-T_0, T_0]$ can be expanded to become a periodic signal $x_p(t)$ with ***fundamental period***: $T > 2T_0$.

- If the period tends to infinity $T \rightarrow \infty$, the periodic signal tends to $x(t)$:

$$\lim_{T \to \infty} x_p(t) = x(t).$$



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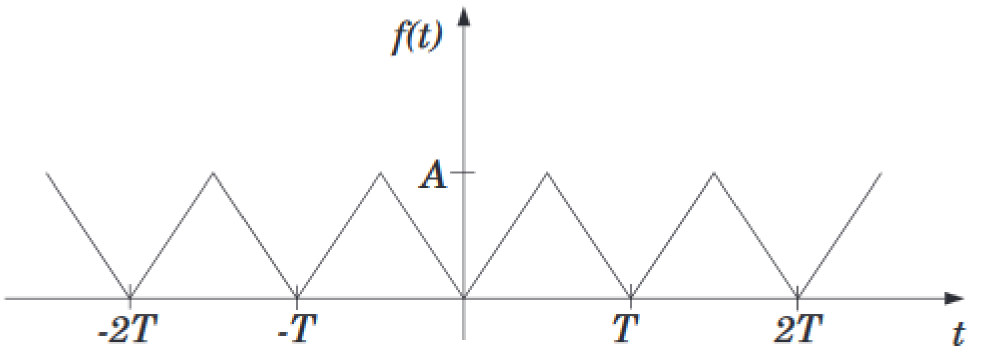
Triangular Fourier Series

- We have the triangular wave function:
- $f(t) = \begin{cases} -\frac{2At}{T}, -T/2 \le t < 0 \\ \frac{2At}{T}, 0 \le t < T/2 \end{cases}$



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Orthogonal signals

Two discrete-time signals $x(n)$ and $x(n)$ are ***orthogonal*** if:

$$\sum_{m=-\infty}^{\infty} x(n)y(n) = 0.$$

Discrete-time functions $\varphi_k(n)$, $k, n = 0, \dots, N-1$, are ***orthonormal*** if:

$$\sum_{n=0}^{N-1} \varphi_k(n)\varphi_l(n) = \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases}.$$

Discrete Orthogonal Signals

They form an orthonormal basis function system, to be used in function $x(n)$ decomposition:

$$x(n) = \sum_{k=0}^{N-1} X(k)\, \varphi_n(k),$$

where:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, \varphi_k(n).$$

Together they form an ***orthogonal signal transform*** pair.

Discrete Orthogonal Signals

Examples of orthogonal function systems:

- ***Discrete-time complex exponential signals***:

$$\varphi_k(n) = e^{-i\frac{2\pi nk}{N}}, \qquad k, n = 0, \ldots, N-1$$

- ***Discrete-time cosine signals***:

$$\varphi_k(n) = \sqrt{\frac{2}{N}} \cos\frac{(2n+1)k\pi}{2N}, \qquad k, n = 0, \ldots, N-1.$$

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Discrete Fourier Transform

Forward DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

- W_N are the $N-$ th complex roots of unity:

$$W_N = e^{-i\frac{2\pi}{N}}, \qquad W_N^N = 1.$$

Inverse DFT (***IDFT***):

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}.$$

- It is extensively used in digital signal processing/analysis.

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Discrete Cosine Transform

- DCT is used in the JPEG and MPEG standards.
- Forward DCT transform:

$$C(0)=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}x(n)$$

$$C(k)=\sqrt{\frac{2}{N}}\sum_{n=0}^{N-1}x(n)\cos\frac{(2n+1)k\pi}{2N}.$$

- Inverse DCT:

$$x(n)=\frac{1}{\sqrt{N}}C(0)+\sqrt{\frac{2}{N}}\sum_{k=1}^{N-1}C(k)\cos\frac{(2n+1)k\pi}{2N}.$$

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Q & A

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