Laplace Transform summary

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Laplace Transform

- **Definition**
- Properties
- Inverse Laplace Transform
- Analysis of LTI Systems

Definition

Unilateral Laplace transform (***LT***) is of defined by:

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt.$$

Bilateral Laplace transform definition:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt.$$

As $s = \sigma + i\Omega$ is a complex number, LT is a mapping $\mathbb{R} \rightarrow \mathbb{C}$ of a signal on the complex plane.

Poles and Zeros

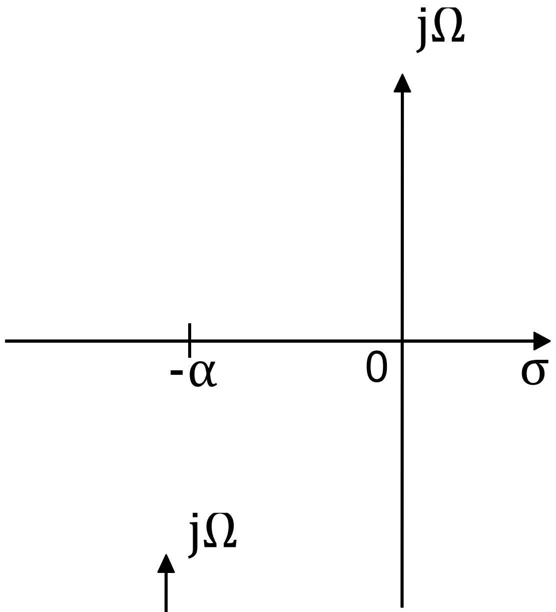
- If LT $X(s)$ is a rational function:

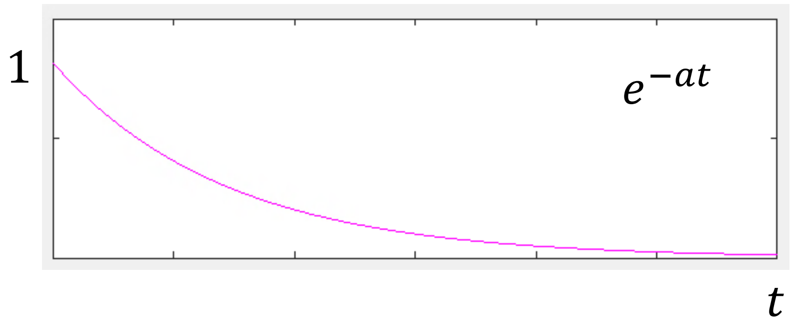
$$X(s) = \frac{N(s)}{D(s)},$$

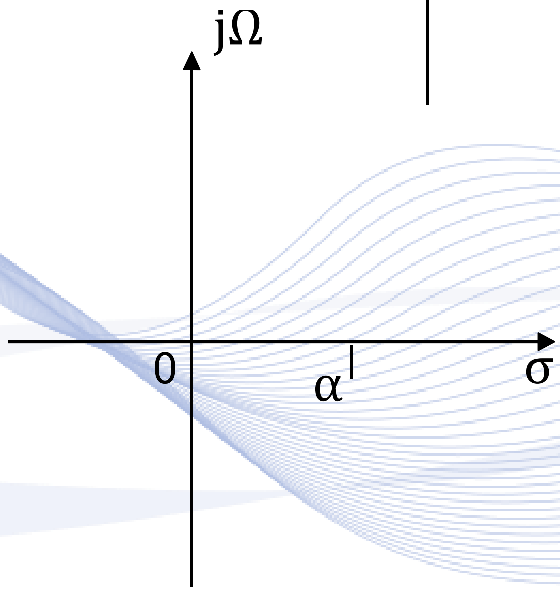
the roots of $D(s)$ are named ***poles*** and the roots of $N(s)$ are named ***zeros***.

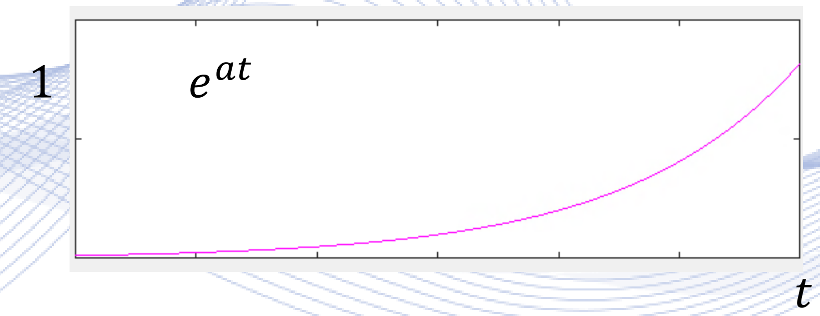
- Poles and zeros can be real of pair of complex conjugate numbers.

Poles and Zeros



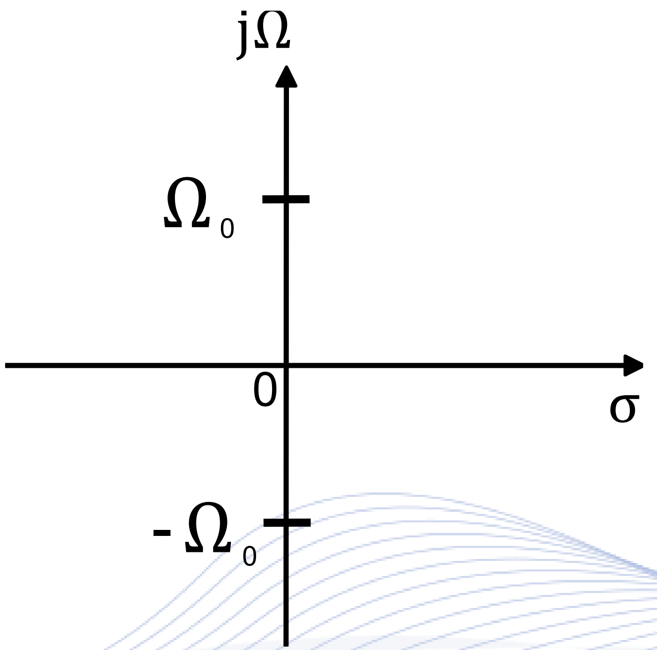


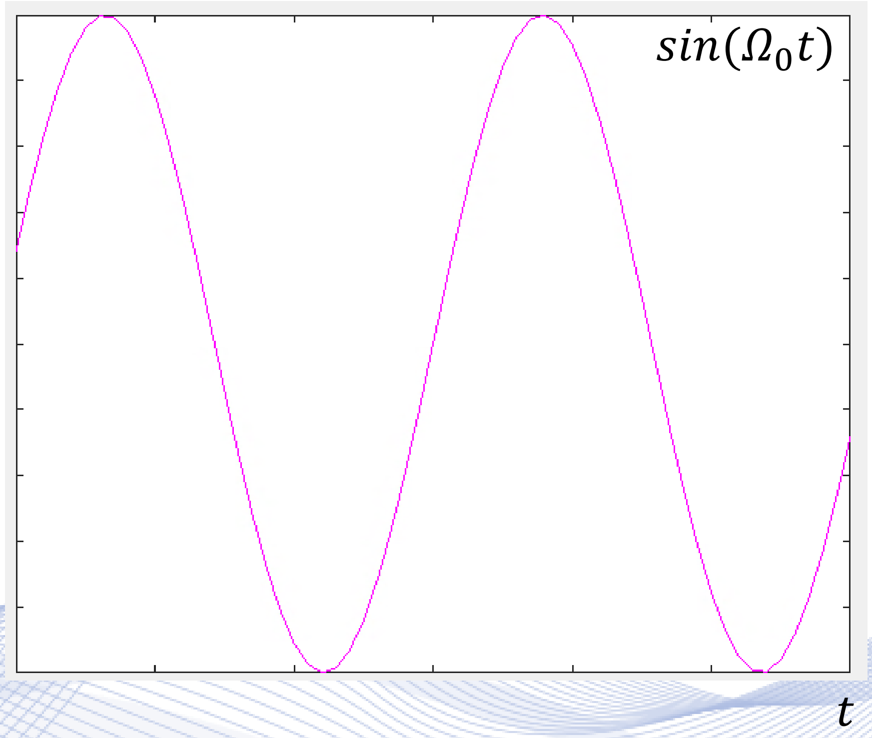




Single poles on the real axis.

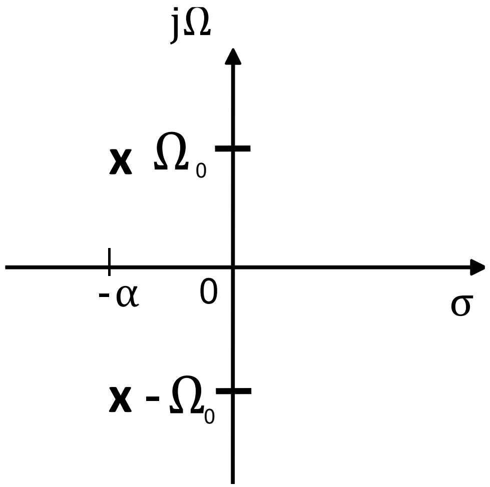
Poles and Zeros



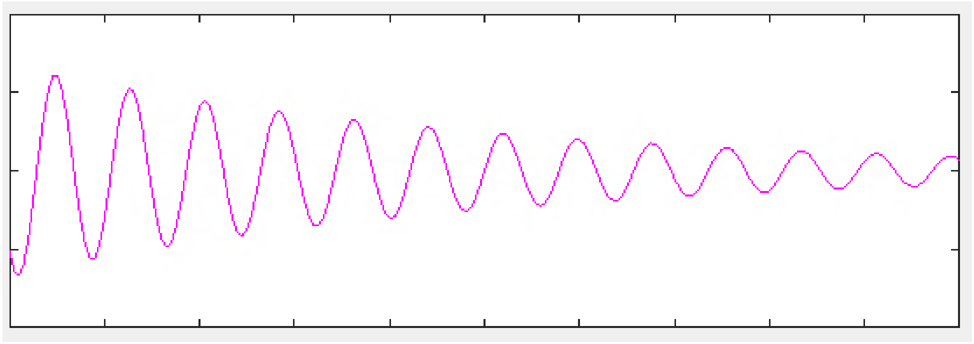


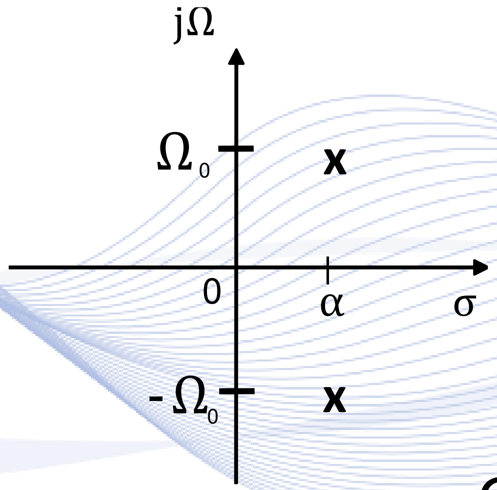
Multiple poles on the imaginary axis.

Poles and Zeros

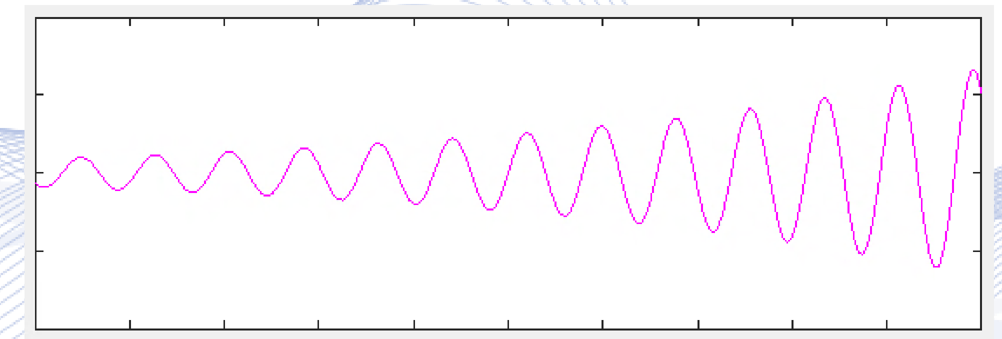


$e^{-at} sin\Omega_0 t$





$e^{at} sin\Omega_0 t$



Complex conjugate poles.

Laplace Transform

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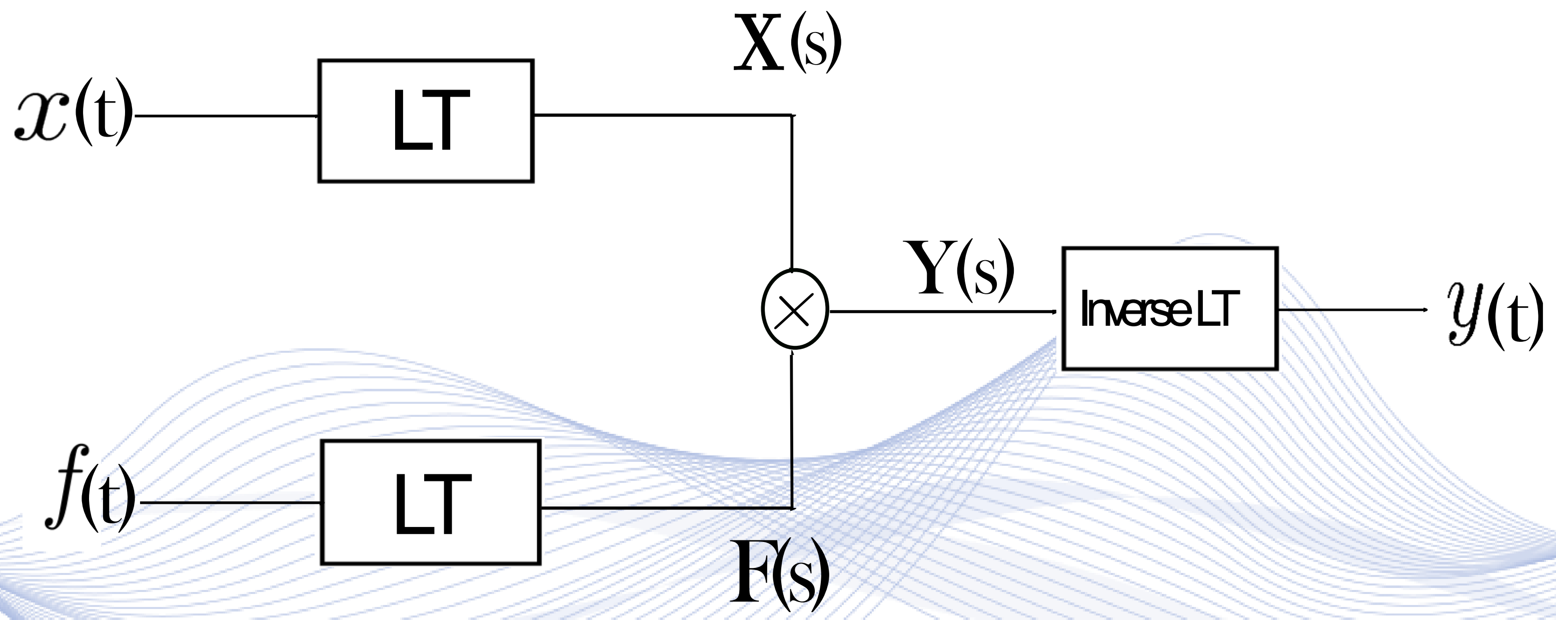
Laplace Transform Properties



Laplace Transform

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- Properties
- **Inverse Laplace Transform**
- Analysis of LTI Systems

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Inverse Laplace Transform

The inverse LT is given by the ***complex inversion integral***:

$$x(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} X(s)e^{st}ds\,, \qquad t > 0.$$

- Parameter σ should lie to the right of all singularities (poles), i.e., $\sigma > \sigma_0$.

Laplace Transform

- Definition
- Properties
- Inverse Laplace Transform calculation
- **Analysis of LTI Systems**

Transfer Function

An LTI system can be described by the convolution of its input $x(t)$ and its ***impulse response*** $h(t), t > 0$:

$$y(t) = \int_0^{\infty} h(\tau)x(t-\tau)d\tau.$$

By applying the convolution property of LT, we have:

$$Y(s) = H(s)X(s).$$

System ***transfer function*** $H(s)$ has a rational form:

$$H(s) = \frac{Y(s)}{X(s)}.$$

Transfer Function

If all the initial conditions are equal to zero, then by applying the LT in both parts of the previous formula, we have:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} = \frac{N(s)}{D(s)}.$$

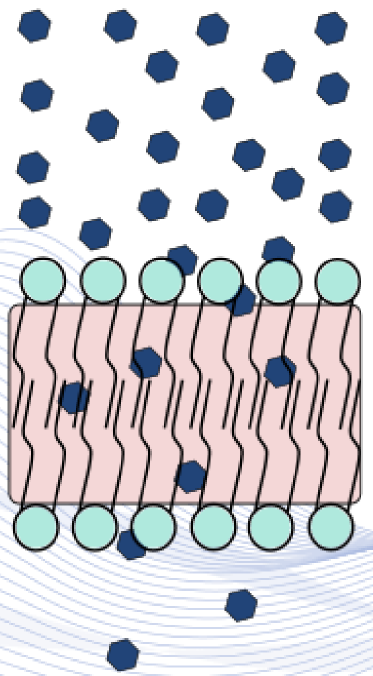
- $N(s), D(s)$ polynomials roots are called ***zeroes*** and ***poles***, respectively.
- As, under certain conditions, $D(s)$ may become zero, leading to system instability, $D(s)$ defines system stability.

Transfer Function

Temporal diffusion system:

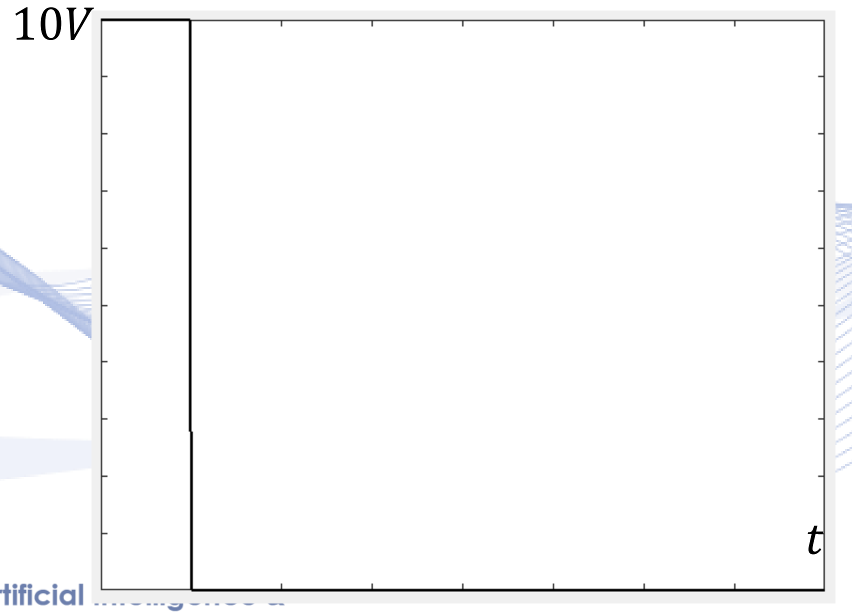
$$\frac{dy(t)}{dt} = c\big(x(t) - y(t)\big).$$

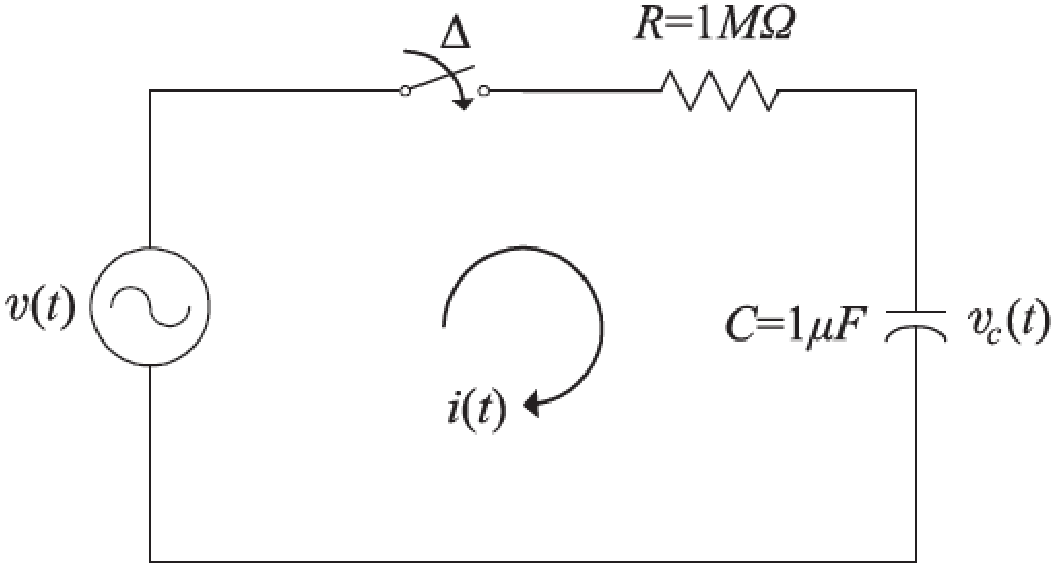
- c: diffusion coefficient.



Transfer Function

RC circuit. The circuit switch turns on at $t = 0$ and the voltage $v(t)$ has the form:

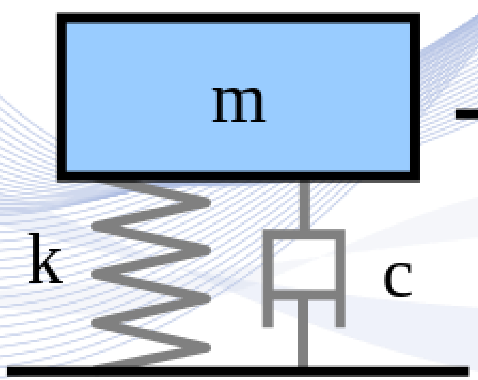




Transfer Function

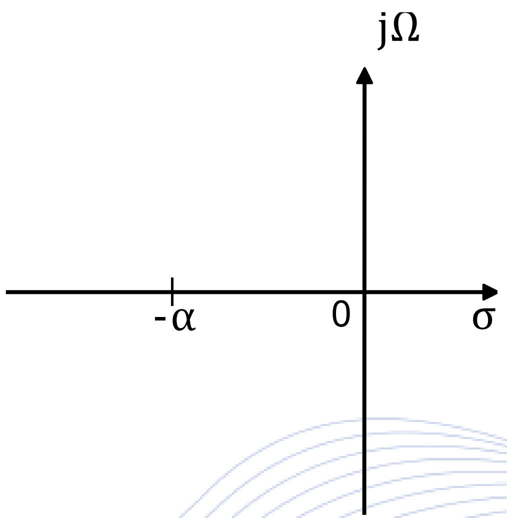
Mass-spring-damper mechanical system. If force $f(t)$ is exercised on a mass m that is attached to a spring having constant k that follows Hooke's law and to a damper having damping constant β, the displacement $y(t)$ is given by the exercised forces:

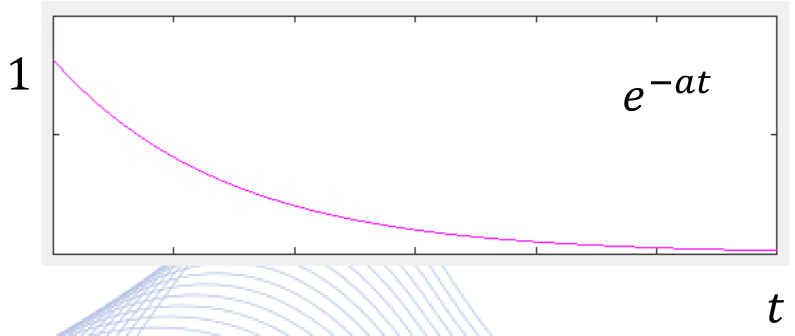
$$m\frac{d^2y(t)}{dt^2} + \beta\frac{dy(t)}{dt} + ky(t) = f(t).$$



- It models car suspension systems.

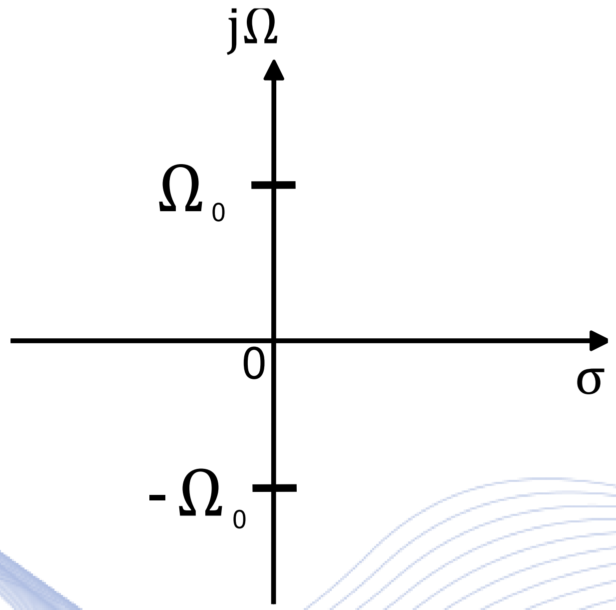
System Stability

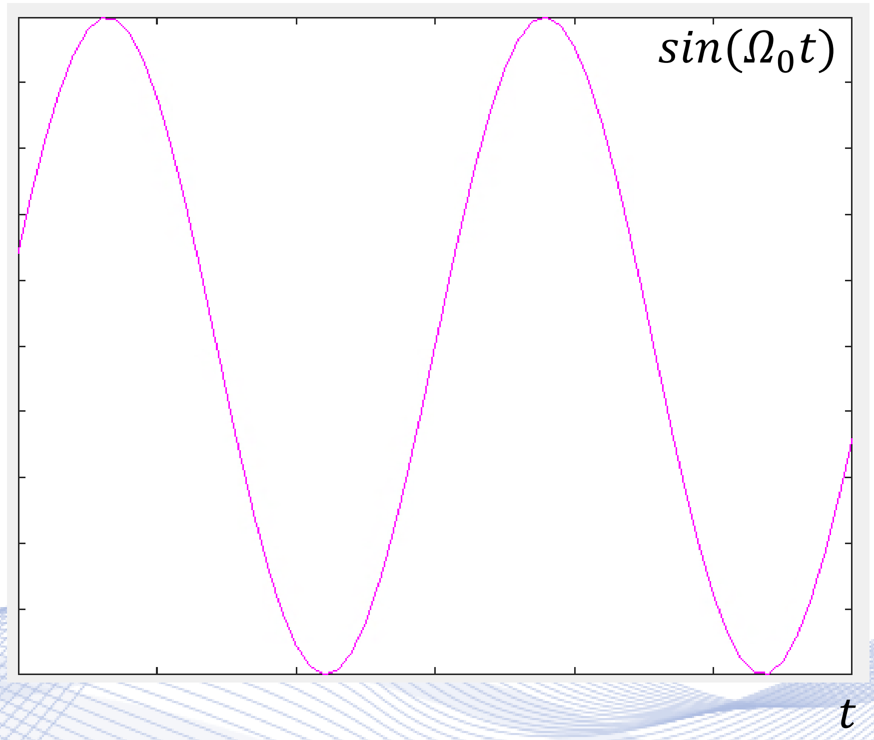




a) The pole is found in the left complex half-plane, thus the system is stable. b) Impulse response.

System Stability





a) The pole is found on the imaginary axis, thus the system oscilates. b) Impulse response.

Bibliography

[OPP2013] A. Oppenheim, A. Willsky, Signals and Systems, Pearson New International, 2013.

[MIT1997] S. K. Mitra, Digital Signal Processing, McGraw-Hill, 1997.

[OPP1999] A.V. Oppenheim, Discrete-time signal processing, Pearson Education India, 1999.

[HAY2007] S. Haykin, B. Van Veen, Signals and systems, John Wiley, 2007.

[LAT2005] B. P. Lathi, Linear Systems and Signals, Oxford University Press, 2005.

[HWE2013] H. Hwei. Schaum's Outline of Signals and Systems, McGraw-Hill, 2013.

[MCC2003] J. McClellan, R. W. Schafer, and M. A. Yoder, Signal Processing, Pearson Education Prentice Hall, 2003.

Bibliography

[PHI2008] C. L. Phillips, J. M. Parr, and E. A. Riskin, Signals, Systems, and Transforms, Pearson Education, 2008.

[PRO2007] J.G. Proakis, D.G. Manolakis, Digital signal processing. PHI Publication, 2007.

[DUT2009] T. Dutoit and F. Marques, Applied Signal Processing. A MATLAB-Based Proof of Concept. New York, N.Y.: Springer, 2009

Bibliography

[PIT2000] I. Pitas, "Digital Image Processing Algorithms and Applications", J. Wiley, 2000.
[PIT2021] I. Pitas, "Computer vision", Createspace/Amazon, in press.
[PIT2017] I. Pitas, "Digital video processing and analysis" , China Machine Press, 2017 (in Chinese).
[PIT2013] I. Pitas, "Digital Video and Television" , Createspace/Amazon, 2013.
[NIK2000] N. Nikolaidis and I. Pitas, "3D Image Processing Algorithms", J. Wiley, 2000.

Q & A

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