Geometric Spaces summary

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Spaces

- **Vector Spaces**
- Affine Spaces
- Metric Spaces

Vector Spaces

A real vector space composed of two binary operations and a set $\mathcal{V}$. $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$: $(\mathbf{u}, \mathbf{v}) \to \mathbf{u} + \mathbf{v}$ and $\mathbb{R} \times \mathcal{V} \to \mathcal{V}$: $(\lambda, \mathbf{v}) \to \lambda\mathbf{v}$ such that [BAE2012] :

- For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$, $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
- For all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- There exists a zero vector $0 \in \mathcal{V}$, i.e., for any $\mathbf{u} \in \mathcal{V}$, $\mathbf{u} + 0 = \mathbf{u}$.
- All $\mathbf{u} \in \mathcal{V}$ has a negative element, i.e., there exists $-\mathbf{u} \in \mathcal{V}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$.

Vector Spaces

- For all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u} \in \mathcal{V}$, $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
- For all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u} \in \mathcal{V}$, $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- For all $\alpha \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- Multiplication by $1 \in \mathbb{R}$ is the identity, i.e., for all $\mathbf{u} \in \mathcal{V}$, $1\mathbf{u} = \mathbf{u}$.

The set $\mathbb{R}$ of real numbers can be exchanged with the set $\mathbb{C}$ of complex numbers and then we get the definition of a complex vector space. We can in fact replace $\mathbb{R}$ with any field, e.g., the set $\mathbb{Q}$ of rational numbers, the set of rational functions, or with finite fields such as $\mathbb{Z}_2 = \{0, 1\}$.

Vector Spaces

For a finite subset $\{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \mathbb{V}$ of a vector space the following three statements are equivalent.

- $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a minimal spanning set.
- $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a maximal linearly independent set.
- Each vector $\mathbf{v} \in \mathbb{V}$ can be expressed as a ***one-of-a-kind*** linear combination $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$.

If $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ both satisfy these conditions then $m = n$.

Vector Spaces

Euclidean Vector Spaces and Symmetric Maps

Euclidean vector space is a real vector space $\mathbb{V}$ equipped with a positive definite, symmetric, bilinear mapping $\mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$: $(\mathbf{u}, \mathbf{v}) \rightarrow \mathbf{u}^T\mathbf{v}$ called the ***inner product***, we have the following:

- For all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, $\mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$.
- For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$, $(\mathbf{u} + \mathbf{v})^T\mathbf{w} = \mathbf{u}^{\mathrm{T}}\mathbf{w} + \mathbf{v}^T\mathbf{w}$.
- For all $\alpha \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, $\alpha(\mathbf{u}^T\mathbf{v}) = (\alpha\mathbf{u}^T)\mathbf{v}$.
- For all $\mathbf{u} \in \mathbb{V}$, $\mathbf{u}^T\mathbf{u} \geq 0$.
- For all $\mathbf{u} \in \mathbb{V}$, $\mathbf{u}^T\mathbf{u} = 0 \iff \mathbf{u} = 0$.

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Vector Spaces

The norm of a vector $\mathbf{u} \in \mathbb{V}$ in Euclidean vector space $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ is defined as $||\mathbf{u}|| = \sqrt{\mathbf{u}^T\mathbf{v}}$. The ***Cauchy–Schwartz*** inequality is a crucial property of an arbitrary inner product.

If $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ is Euclidean vector space then the inner product satisfies the Cauchy–Schwartz inequality $\mathbf{u} \leq \mathbf{uv}, \mathbf{v} \leq \mathbf{uv}$, if and only if one of the vectors ***is a positive multiple of the other***, equality is achieved.

Vector Spaces

Eigenvalues and Eigenvectors

The geometric multiplicity of E is the name given to its dimension. If $\mathbf{u}_1, \dots, \mathbf{u}_m$ is a basis for $\mathbb{V}$, $\mathbf{A}$ is the matrix for L in this basis and a vector $\mathbf{v} \in \mathbb{V}$ has coordinates $\mathbf{v}$ with respect to this basis then $L(\mathbf{v}) = \lambda\mathbf{v} \iff \mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

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Spaces

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- **Affine Spaces**
- Metric Spaces

Affine Spaces

An affine space is a set $\mathcal{X}$ that admits a free transitive action of a vector space $\mathbb{V}$ [BAE2012] . That is, there is a map $\mathcal{X} \times \mathbb{V} \rightarrow \mathcal{X}$: $(\mathbf{x}, \mathbf{v}) \rightarrow \mathbf{x} + \mathbf{v}$, called ***translation*** by the vector $\mathbf{v}$, such that:

- The inclusion of vectors is referred to as the structure of translations, for all $\mathbf{x} \in \mathcal{X}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{V}, \mathbf{x} + (\mathbf{u} + \mathbf{v}) = (\mathbf{x} + \mathbf{u}) + \mathbf{v}$.

- The identity is represented by the zero vector for all $\mathbf{x} \in \mathcal{X}$,

$$\mathbf{x} + 0 = \mathbf{x}.$$

Affine Spaces

Affine Maps

Specifically, we have the following proposal. Let f be an affine map between two affine spaces $(\mathbb{X}, \mathbb{U})$ and $(\mathbb{Y}, \mathbb{V})$. Then there is a unique ***linear map*** $L: \mathbb{U} \rightarrow \mathbb{V}$ such that $f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x}) + L(\mathbf{v})$ for all $\boldsymbol{x} \in \mathbb{X}$ and $\mathbf{u} \in \mathbb{U}$.

Spaces

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- **Metric Spaces**

Metric Spaces

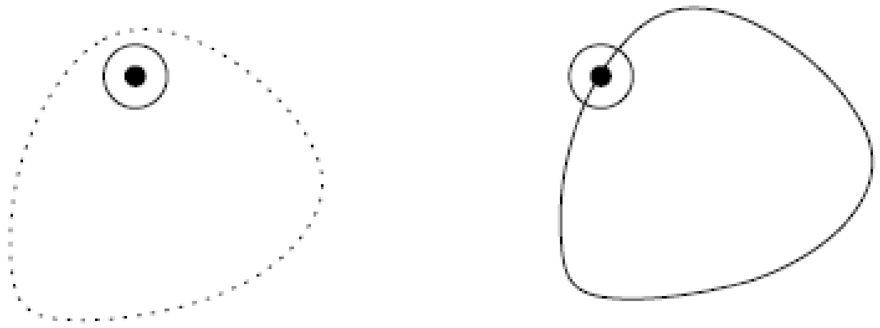
A metric space $(\mathcal{X}, d)$ is a set $\mathcal{X}$ equipped with a map d [BAE2012]:$\mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ that meets the following three criteria:

- Symmetry, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$: $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.
- The triangle inequality, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$: $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.
- Positivity, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$: $d(\boldsymbol{x}, \mathbf{y}) \geq 0, d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$.

Let $(\mathcal{X}, d)$ be a metric space. The ***open ball*** with radius $r > 0$ and center $\mathbf{x} \in \mathcal{X}$is the set $B(\mathbf{x}, r) = \{\mathbf{y} \in \mathcal{X} | \, d(\mathbf{x}, \boldsymbol{y}) < r\}$.

Metric Spaces

Assume that $\mathbb{X}$ is a metric space. A subset $\mathcal{U} \subseteq \mathbb{X}$ is called an *open set* if there for all points $\mathbf{x} \in \mathcal{U}$ exists an open ball $B(\mathbf{x}, r) \subseteq \mathcal{U}$.



An open set to the left with space for a ball around each point. Every ball around a point on the boundary is not found in a non open set to the right.

Q & A

Thank you very much for your attention!

More material in
http://icarus.csd.auth.gr/cvml-web-lecture-series/

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