Discrete-time Signals and Systems Summary

N. Bravos, Prof. Ioannis Pitas
Aristotle University of Thessaloniki
pitas@csd.auth.gr
www.aiia.csd.auth.gr
Version 1.6

Discrete-time Signals and Systems

- Discrete-time Signals
- Discrete-time Systems
- 1D Convolutions
- FIR and IIR Systems

Discrete-time Signals

A discrete-time signal is mathematically represented as a sequence of data samples (typically real numbers), in which the n-th sequence sample is denoted by $x(n)$:

$$\{\, x(n) \,\}, \quad -\infty < n < \infty,$$

where n is an integer index.

A discrete-time signal $x(n)$ is a function of the form:

$$x(n): \quad \mathbb{Z} \rightarrow \mathbb{R}.$$

Complex signal: $x(n): \quad \mathbb{Z} \rightarrow \mathbb{C}$.

Discrete-time Signals

A digital signal $x(n)$ is a discrete-time function of the form:

$$x(n): \quad \mathbb{Z} \rightarrow [0, \dots, 2^B - 1],$$

when signal sample has B bits per sample.

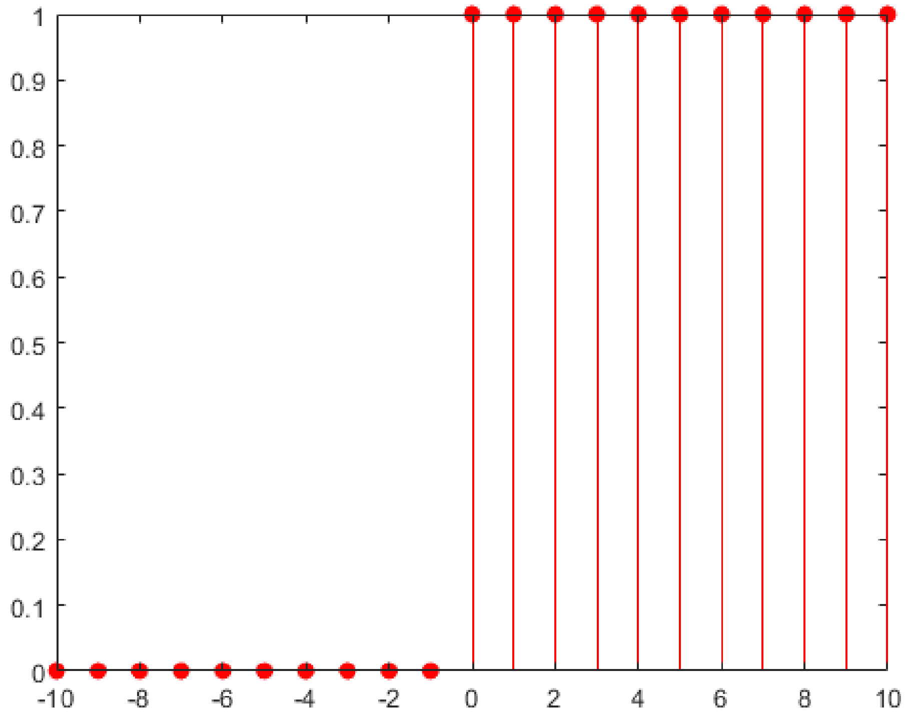
- If $B = 8$, each signal sample has 256 distinct values.

Unit step signal

Formula

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

Graph



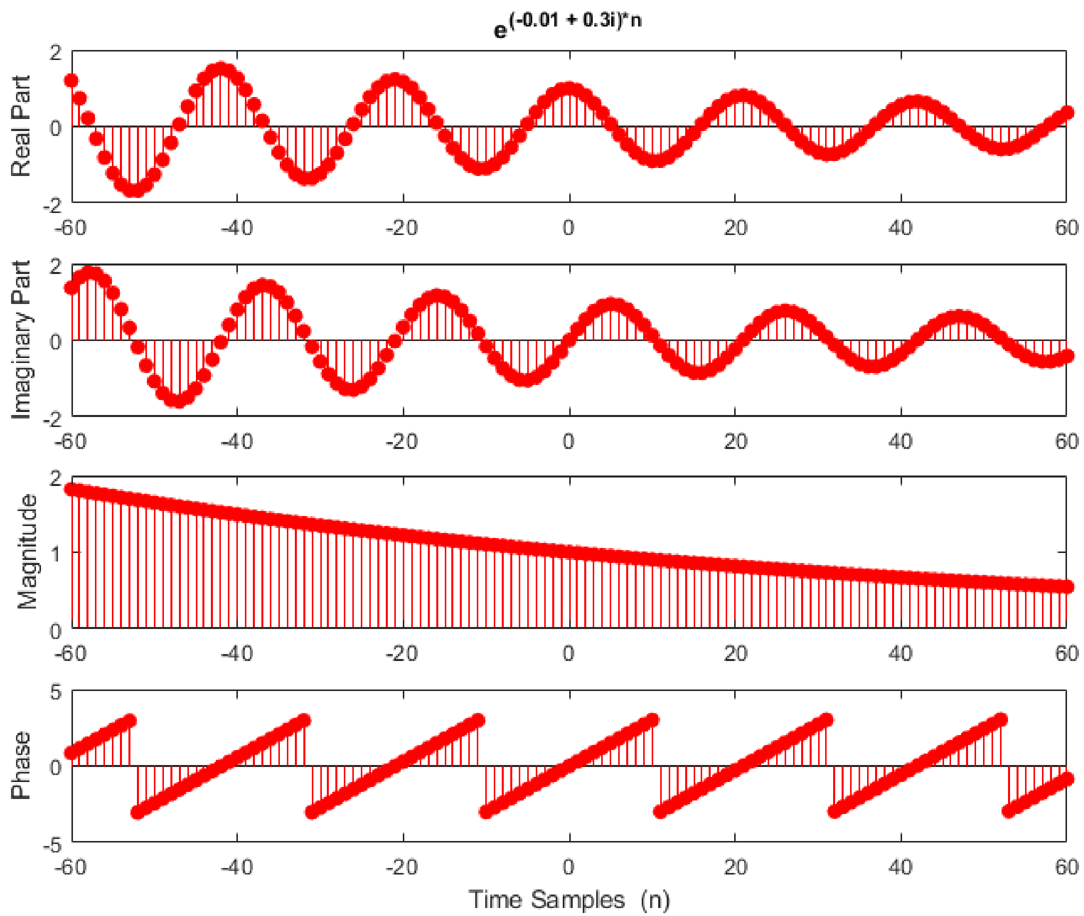
Complex Exponential Signal

Formula

$$x(n) = ae^{i\omega n} = a\cos(\omega n) + ia\sin(\omega n).$$

- a: complex signal ***magnitude***.
- $\varphi(\omega) = \omega n$: complex signal ***phase***.

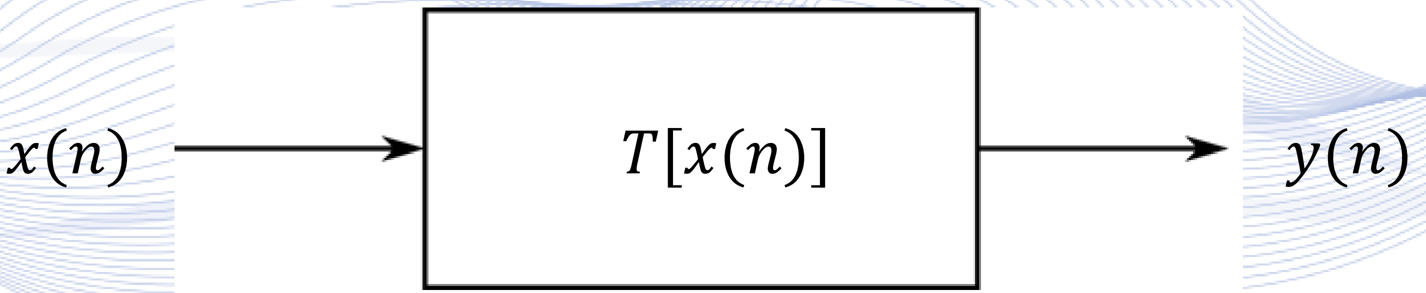
Graph



Discrete-time Systems

A ***discrete-time system*** is defined mathematically as a transformation that maps an input sequence $x(n)$ into an output sequence $y(n)$:

$$y(n) = T[x(n)].$$



1D Linear Systems

Linear Shift-Invariant (LSI) systems:

- Linearity:

$$T[ax_1 + bx_2] = aT[x_1] + bT[x_2].$$

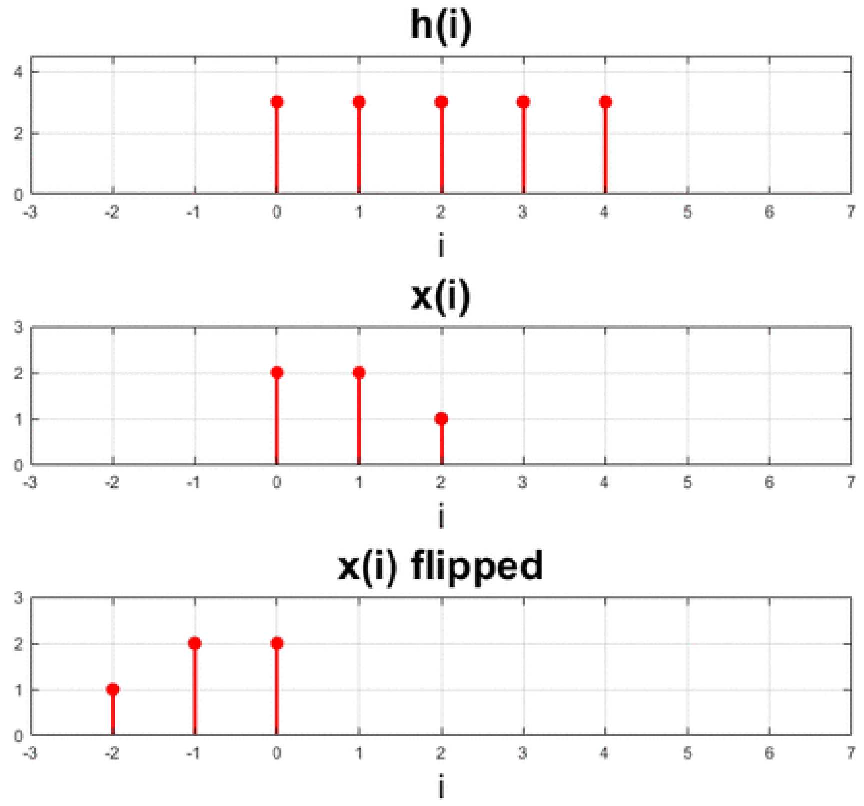
- Shift-Invariance:

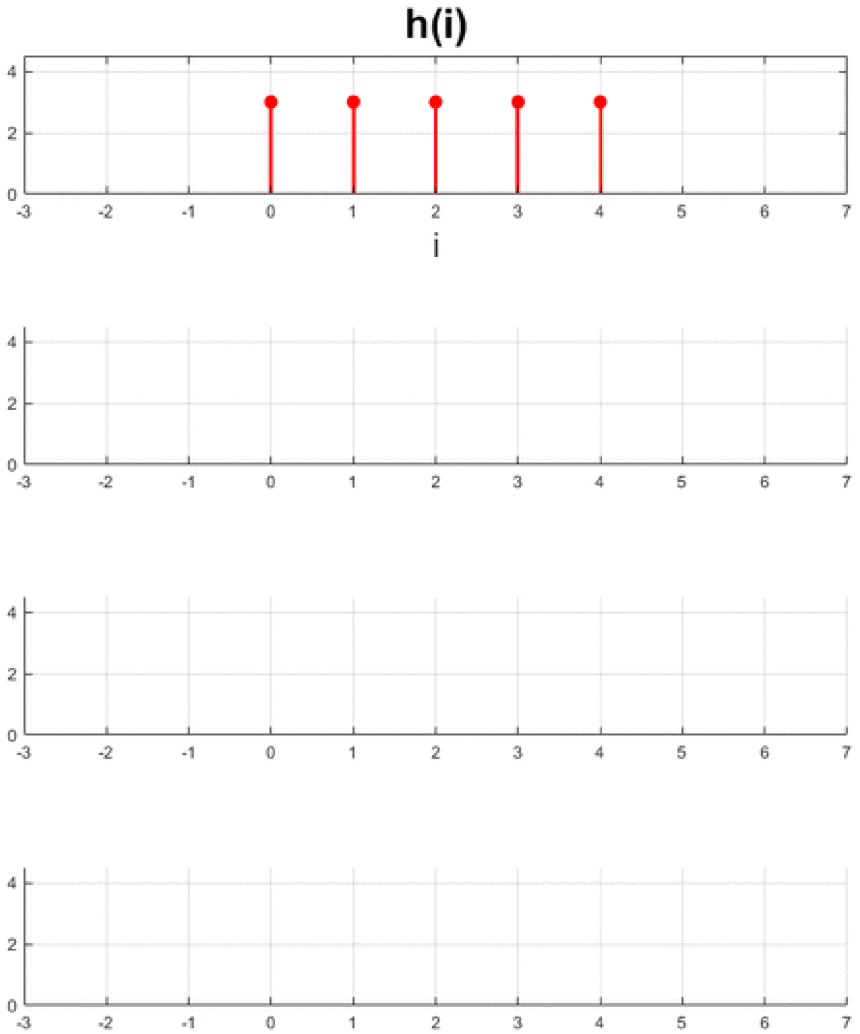
$$y(n) = T[x(n)] \Rightarrow y(n-m) = T[x(n-m)].$$

- LSI system is defined by its ***impulse response function*** $h(n)$ and ***convolution*** operator:

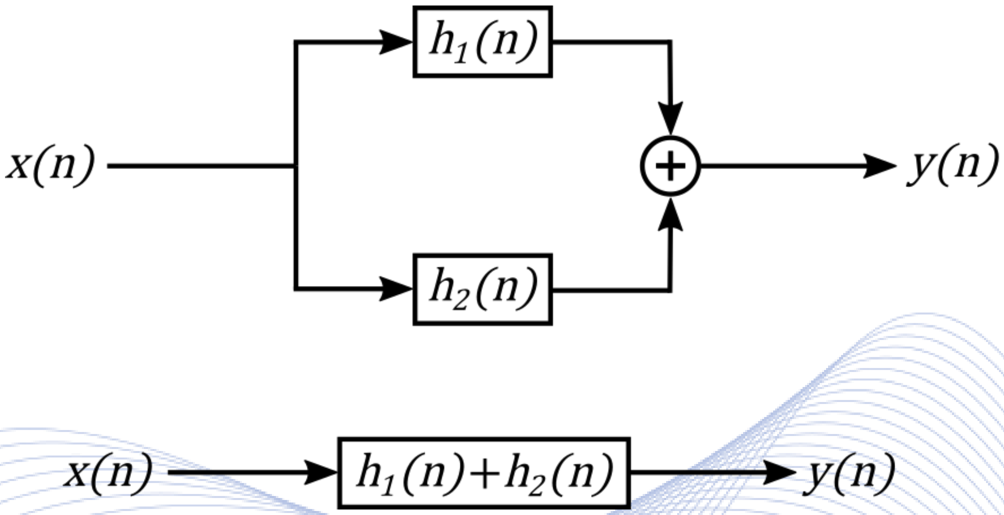
$$y(n) = h(n) * x(n).$$

Linear 1D convolution – Example



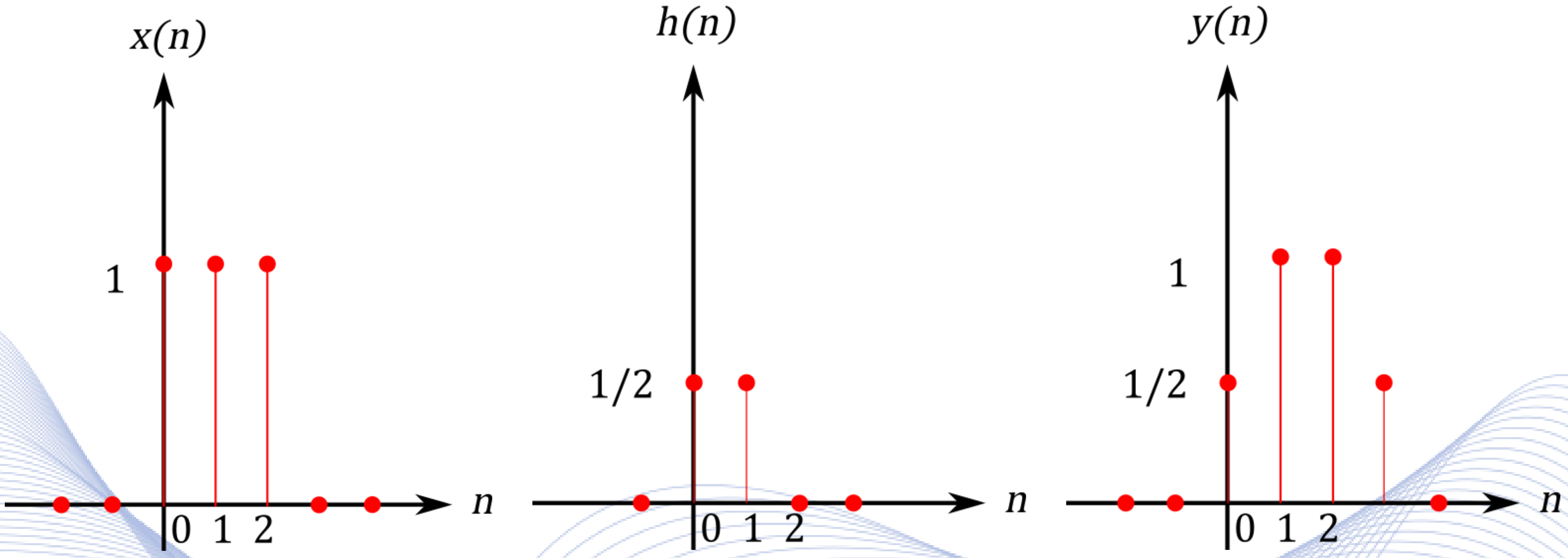


Convolution Properties



LTI System distributivity.

Convolution Example 1



$$y(n) = \sum_{i=0}^{1} h(i)x(n-i) = \frac{1}{2}\big(x(n) + x(n-1)\big).$$

$$y(n) = 0, \qquad n \leq -1$$

$$y(0) = \frac{1}{2}\big(x(0) + x(-1)\big) = \frac{1}{2}$$

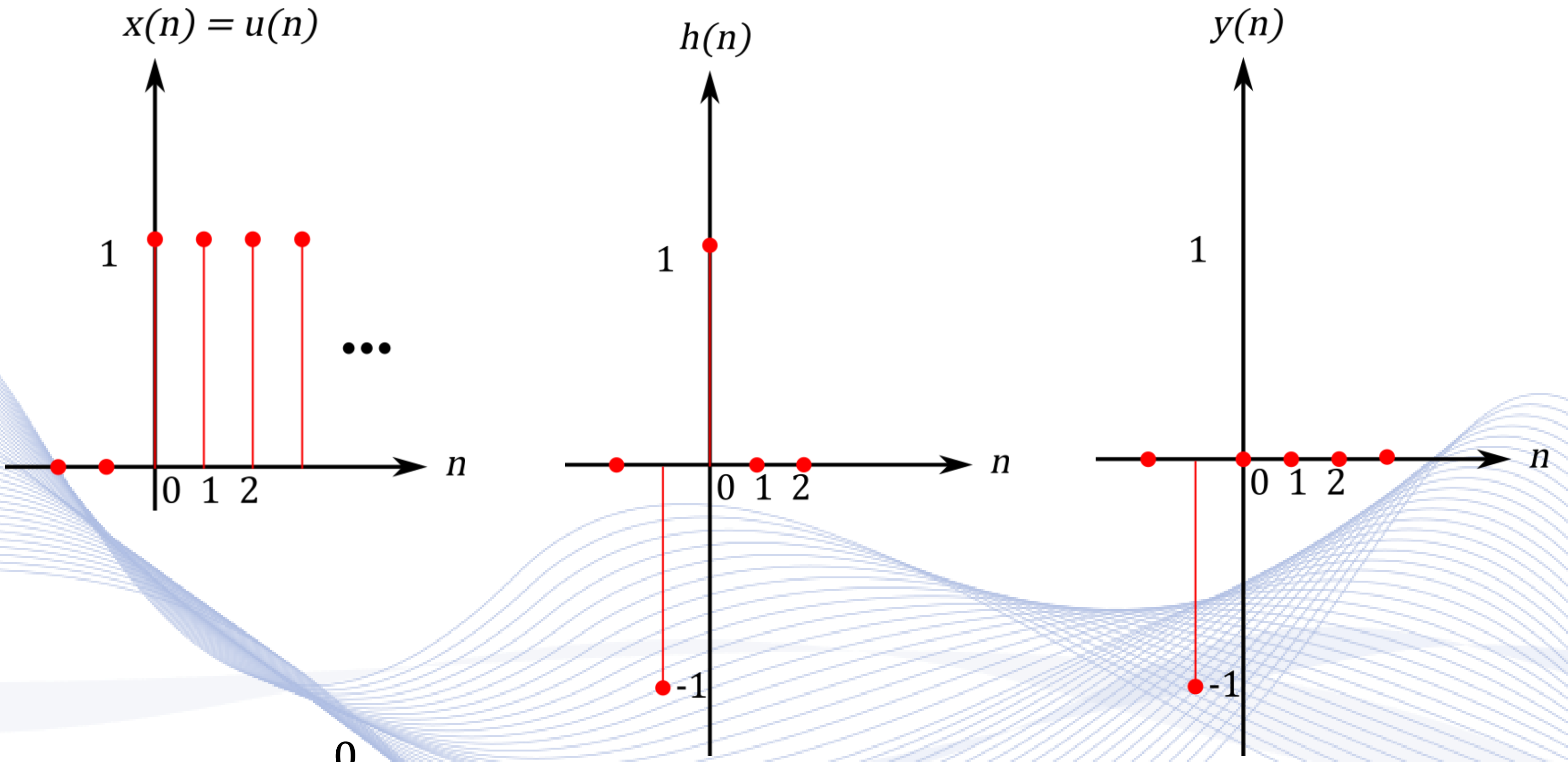
$$y(1) = \frac{1}{2}\big(x(1) + x(0)\big) = 1$$

$$y(2) = \frac{1}{2}\big(x(2) + x(1)\big) = 1$$

$$y(3) = \frac{1}{2}\big(x(3) + x(2)\big) = \frac{1}{2}$$

$$y(n) = 0, \qquad n \geq 4$$

Convolution Example 6



$$y(n) = \sum_{i=-1}^{0} h(i)x(n-i) = h(-1)x(n+1) + h(0)x(n) = x(n) - x(n+1).$$

Convolution Example 7

Calculate the output of a LTI system whose impulse response is:

$$h(n) = \begin{cases} 1, & 0 \leq n \leq M-1 \\ 0, & n < 0, \quad n > M-1. \end{cases}$$

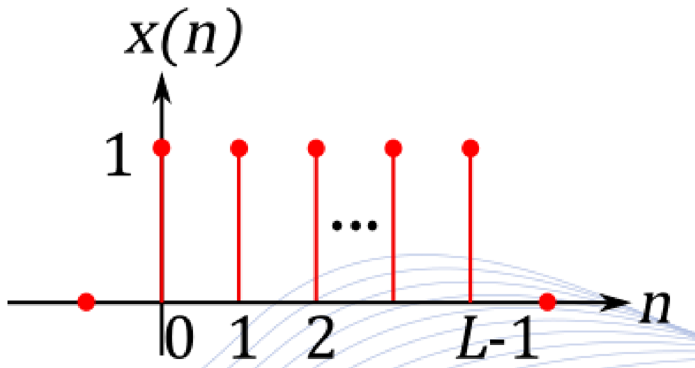
while the input is:

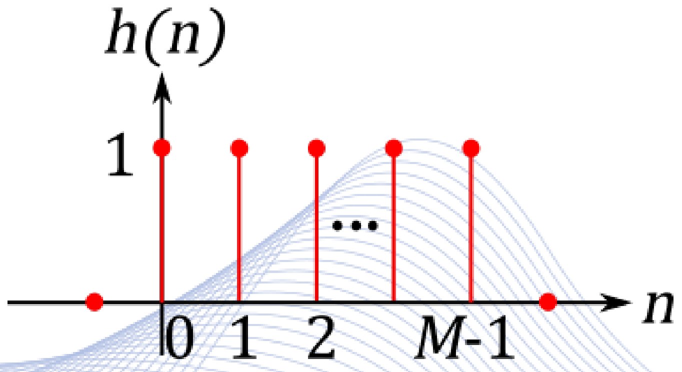
$$x(n) = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & n < 0, \quad n > L-1. \end{cases}$$

where $M < L$.

Convolution Example 7

- We have to convolute two rectangular pulses.
- Input duration is L, while impulse response duration is M samples.





- It is given that $M < L$.
- We can conclude that the output will be a trapezoid because the pulses have unequal duration.

Linear 1D correlation

- Correlation of template signal h and input signal $x(n)$ (inner product):

$$r_{hx}(n) = \sum_{i=0}^{M-1} h(i)x(n+i) = \mathbf{h}^T\mathbf{x}(n).$$

- $\mathbf{h} = [h(0), \ldots, h(M-1)]^T$: template vector.
- $\mathbf{x}(n) = [x(n), \ldots, x(n+M-1)]^T$: local signal vector.

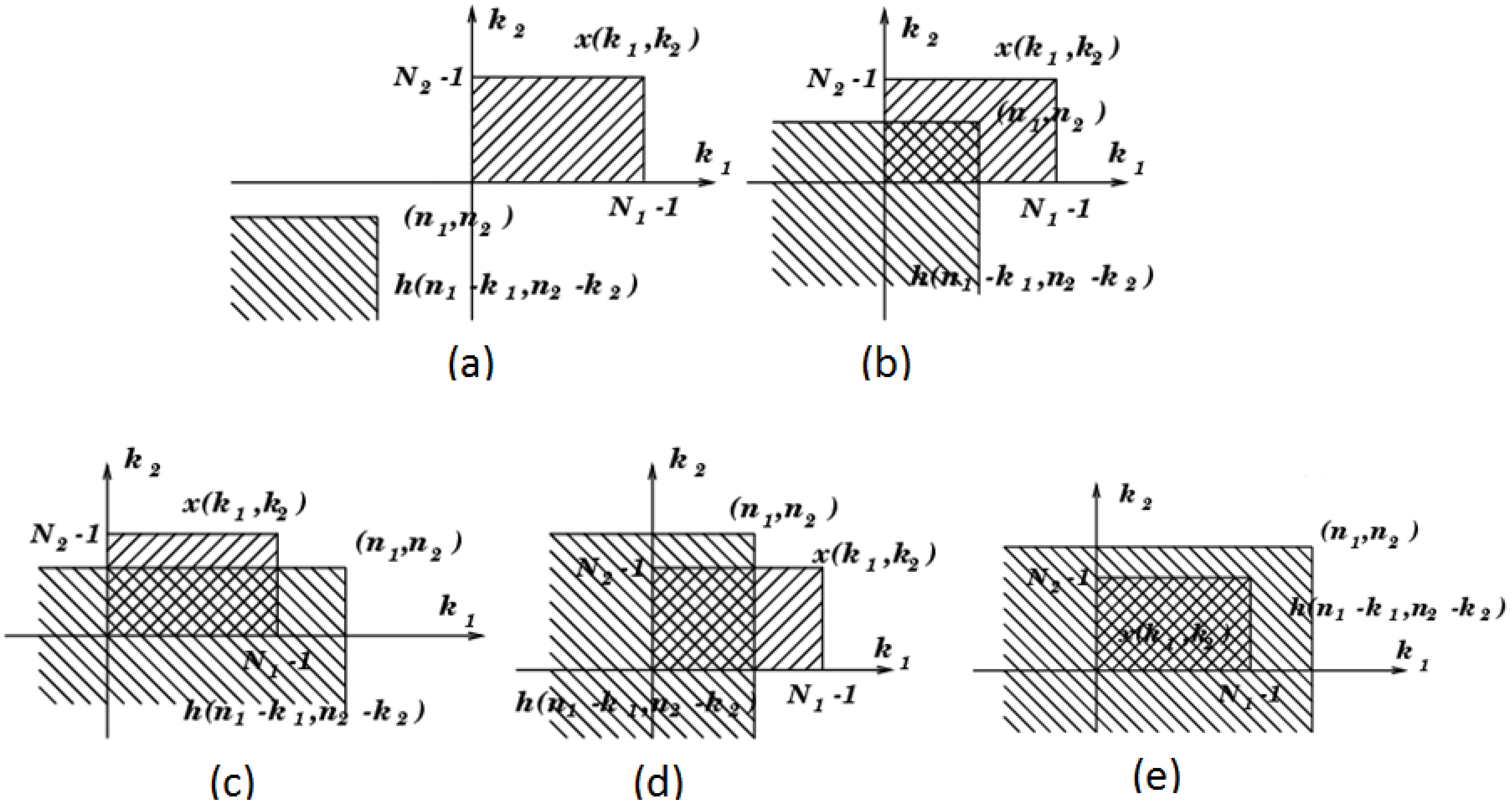
2D convolutions

- A 2D linear shift invariant system is described by a 2D convolution of input x with a convolutional kernel h:

$$y(n_1, n_2) = h(n_1, n_2) ** x(n_1, n_2) = \sum_{i_1} \sum_{i_2} h(i_1, i_2) x(n_1 - i_1, n_2 - i_2).$$

- Input x has typically limited region of support (size), e.g., it can be an image of $M_1 \times M_2$ pixels.
- Convolutional kernel h may have limited or infinite region of support.

2D convolutions



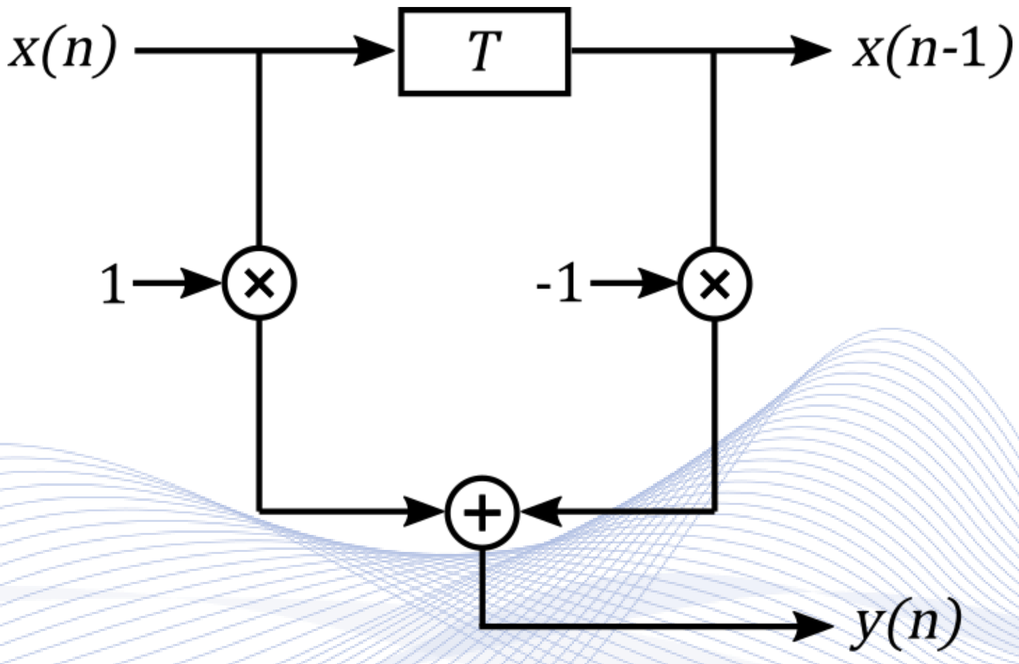
Visualization of 2D convolution calculation.

FIR and IIR Systems

- If a system has an impulse response of finite length $h(i), i = 1, \ldots, M$, it is called ***Finite Impulse Response (FIR)*** system:

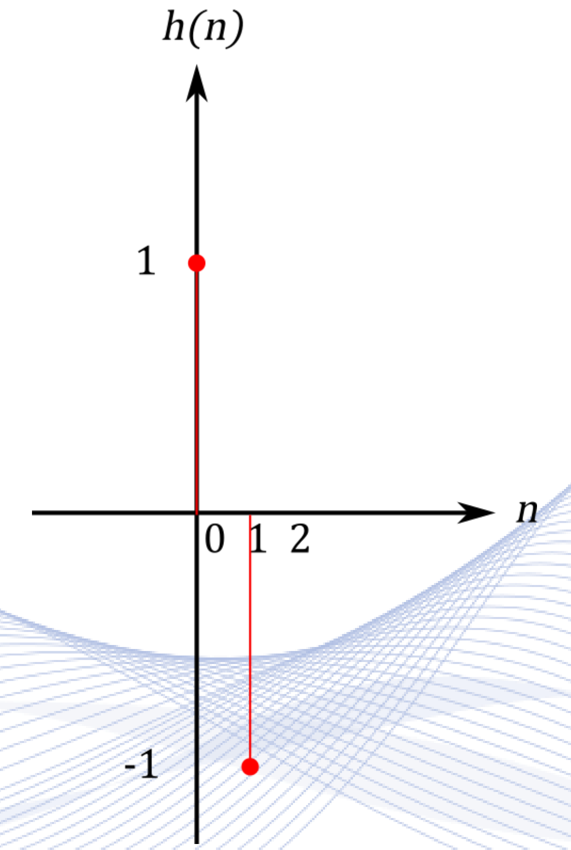
$$y(n) = h(n) * x(n) = \sum_{i=0}^{M-1} h(i)x(n-i).$$

Numerical differentiator



$$y(n) = x(n) - x(n-1).$$

Numerical differentiator



$$y(n) = \sum_{i=0}^{1} h(n)x(n-i) = x(n) - x(n-1).$$

IIR Systems

If a system has an impulse response of infinite length $h(i), i = -\infty, \ldots, \infty$, is referred as an ***Infinite Impulse Response (IIR)*** system.

An IIR system is described by a ***difference equation***:

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k),$$

having coefficients $a_k, k = 0, \ldots, N$ and $b_k, k = 0, \ldots M$.

IIR Systems

An IIR system can also be described by a ***recursive*** formula:

$$y(n) = \frac{1}{a_0}\left(\sum_{i=0}^{M} b_i x(n-i) - \sum_{i=1}^{N} a_i y(n-i)\right).$$

- Output $y(n)$ is given at time n in terms of the present input sample and the previous values of the input and output samples.

Bibliography

[OPP2013] A. Oppenheim, A. Willsky, Signals and Systems, Pearson New International, 2013.

[MIT1997] S. K. Mitra, Digital Signal Processing, McGraw-Hill, 1997.

[OPP1999] A.V. Oppenheim, Discrete-time signal processing, Pearson Education India, 1999.

[HAY2007] S. Haykin, B. Van Veen, Signals and systems, John Wiley, 2007.

[LAT2005] B. P. Lathi, Linear Systems and Signals, Oxford University Press, 2005.

[HWE2013] H. Hwei. Schaum's Outline of Signals and Systems, McGraw-Hill, 2013.

[MCC2003] J. McClellan, R. W. Schafer, and M. A. Yoder, Signal Processing, Pearson Education Prentice Hall, 2003.

Bibliography

[PHI2008] C. L. Phillips, J. M. Parr, and E. A. Riskin, Signals, Systems, and Transforms, Pearson Education, 2008.

[PRO2007] J.G. Proakis, D.G. Manolakis, Digital signal processing. PHI Publication, 2007.

[DUT2009] T. Dutoit and F. Marques, Applied Signal Processing. A MATLAB-Based Proof of Concept. New York, N.Y.: Springer, 2009

Bibliography

[PIT2000] I. Pitas, “Digital Image Processing Algorithms and Applications”, J. Wiley, 2000.
[PIT2021] I. Pitas, “Computer vision”, Createspace/Amazon, in press.
[PIT2017] I. Pitas, “Digital video processing and analysis” , China Machine Press, 2017 (in Chinese).
[PIT2013] I. Pitas, “Digital Video and Television” , Createspace/Amazon, 2013.
[NIK2000] N. Nikolaidis and I. Pitas, “3D Image Processing Algorithms”, J. Wiley, 2000.

Q & A

Thank you very much for your attention!

More material in
http://icarus.csd.auth.gr/cvml-web-lecture-series/

Contact: Prof. I. Pitas
pitas@csd.auth.gr