Continuous-time Signals and Systems Summary

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Information Analysis Lab

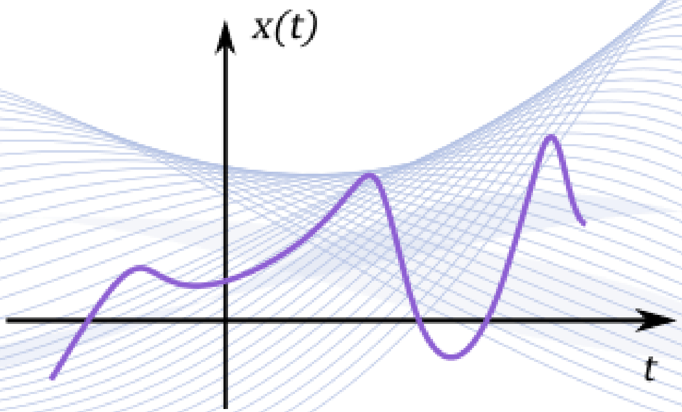
Continuous-time Signals and Systems

- **Continuous-time Signals**
  - Signal Operations
  - Important continuous-time signals
- Continuous-time LTI Systems
  - LTI System response, Convolution, Correlation
  - Properties of LTI Systems
  - Eigenfunctions of LTI Systems
  - Standard Differential Equation for LTI Systems

Continuous-time Signals

A function of a continuous variable (typically real variable) is called continuous-time signal $x(t)$, where $t \in \mathbb{R}$.

- Typically, t denotes time.

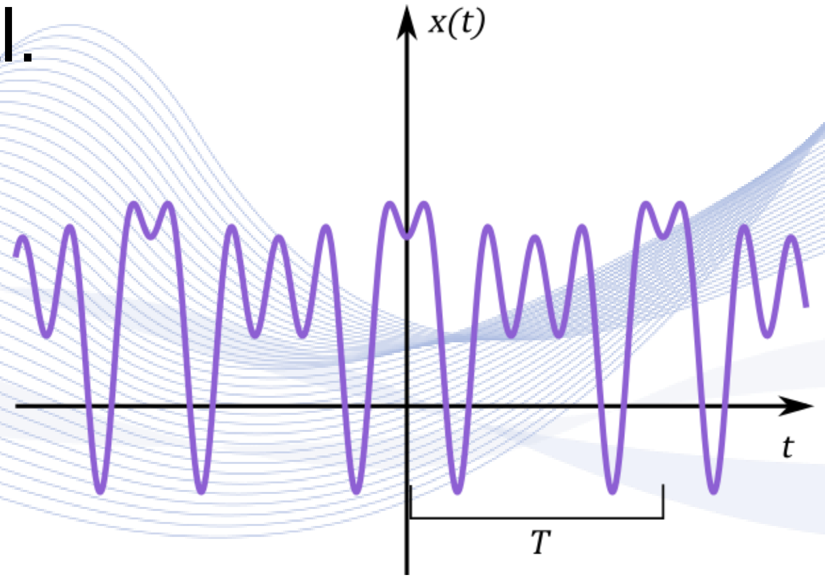


Continuous-time Signals

A continuous-time signal $x(t)$ is ***periodic***, when there is a positive non-zero value T for which:

$$x(t + T) = x(t) \qquad \text{for all } t.$$

- T is referred to as the period of the signal.



- ***Frequency***: $F = \frac{1}{T}$. It is measured in Hertz (Hz).
- ***Angular frequency***: $\Omega = 2\pi F = \frac{2\pi}{T}$.

Important continuous-time signals

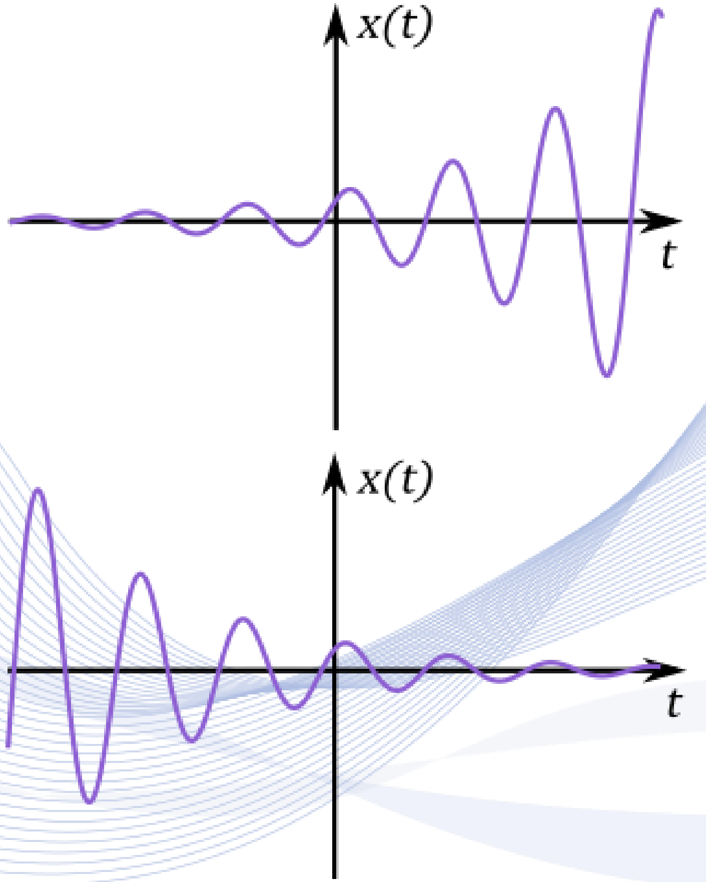
- Trigonometric signals
- Complex exponential signal
- Real exponential signal
- Unit impulse signal
- Unit step signal

Important continuous-time signals

Complex exponential signal:

$$x(t) = \mathrm{e}^{st} = e^{(\sigma + i\Omega t)} = e^{\sigma t}\,(\cos\Omega t + i\sin\Omega t).$$

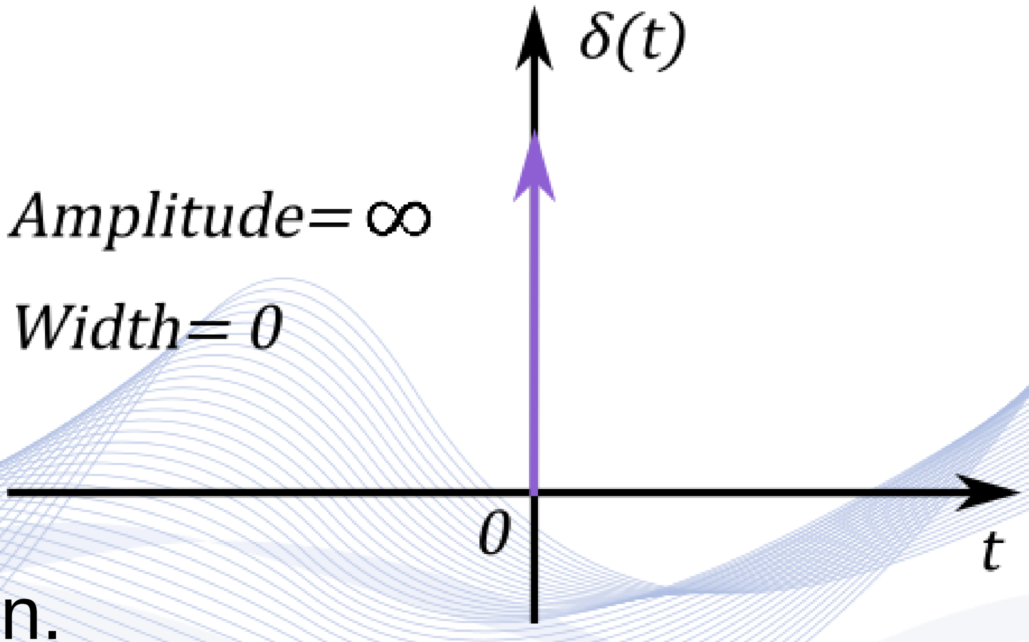
- Its real and imaginary part are exponentially increasing $(\sigma > 0)$ or decreasing $(\sigma < 0)$ sinusoidal signals.



Important continuous-time signals

Unit impulse signal:

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0. \end{cases}$$

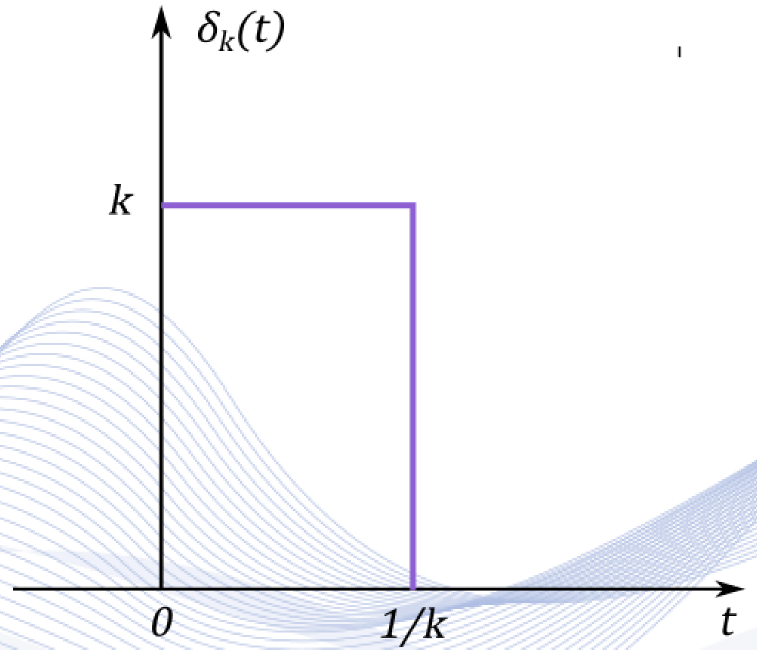


- Also known as ***Dirac delta*** function.
- It is a ***generalized function***.

Important continuous-time signals

Delta function definition as a limit of a sequence of functions $\delta_k(t)$:

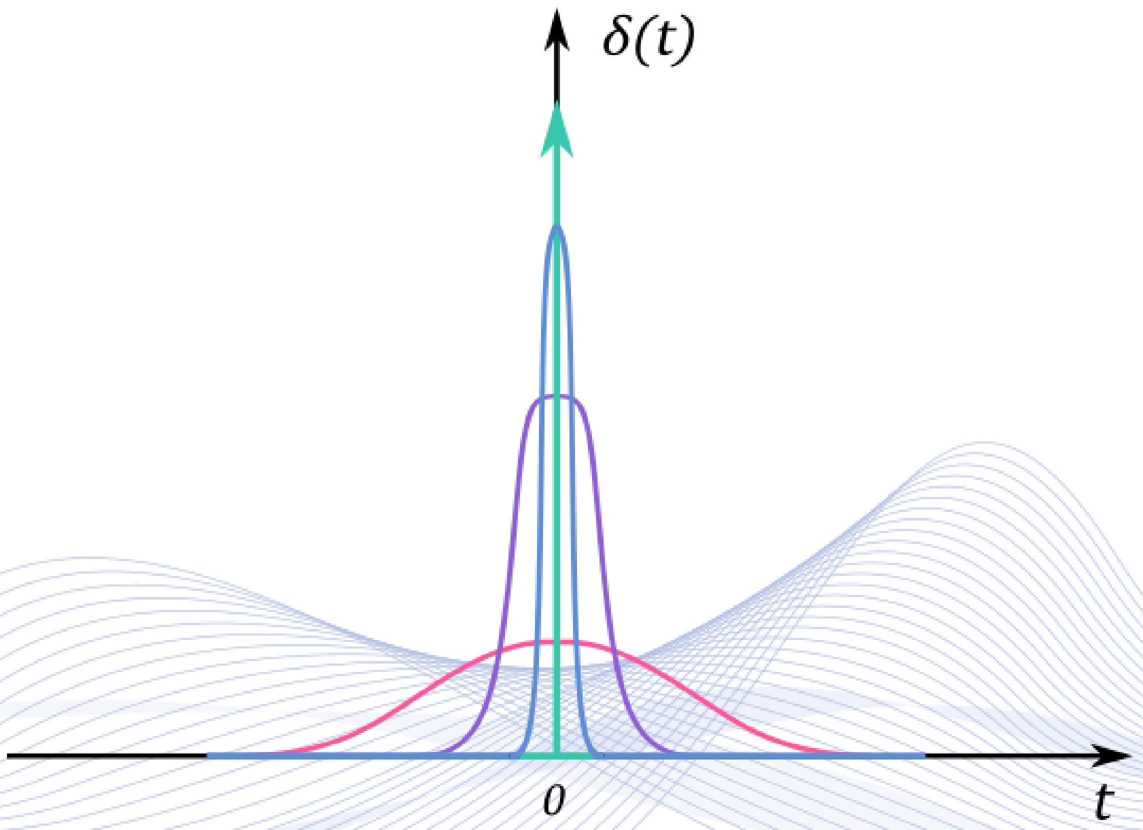
$$\delta_k(t) = \begin{cases} 0\,, & t \geq \frac{1}{k} \\ k, & 0 < t < \frac{1}{k} \\ 0\,, & t < 0. \end{cases}$$



$$\lim_{k \to \infty} \delta_k(t) = \delta(t).$$

- Delta function can also be defined as a limit of a sequence of exponential or sync functions.

Important continuous-time signals



Delta function defined as the limit of a series of Gaussian functions with varying variance.

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Continuous-time Systems

Continuous input

Continuous output

System definition: a transformation of input signal $x(t)$ into output signal $y(t)$:

$$y(t) = T[x(t)].$$

Responses of LTI Systems

Impulse Response and convolution:

When the input of an LTI system is the delta function $\delta(t)$, its output is called ***impulse response*** $h(t)$.

LTI system output is a ***convolution*** of input signal $x(t)$ and impulse response $h(t)$:

$$y(t) \triangleq x(t) * h(t) \triangleq \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau.$$

- $h(t)$ is flipped about $\tau = 0$ and shifted to the right by t.

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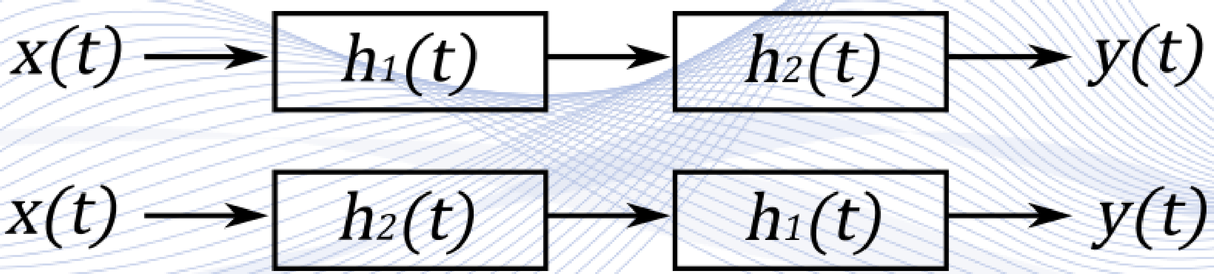
Convolution

Convolution Properties:

- LTI system output to delta function input:

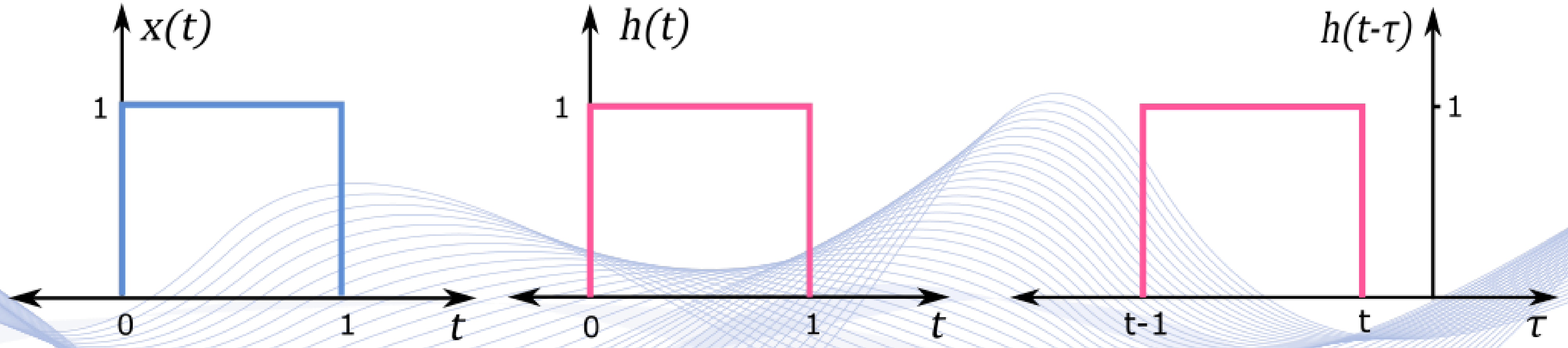
$$y(t) = \delta(t) * h(t) = h(t).$$

- ***Commutativity***: $h_1(t) * h_2(t) = h_2(t) * h_1(t)$



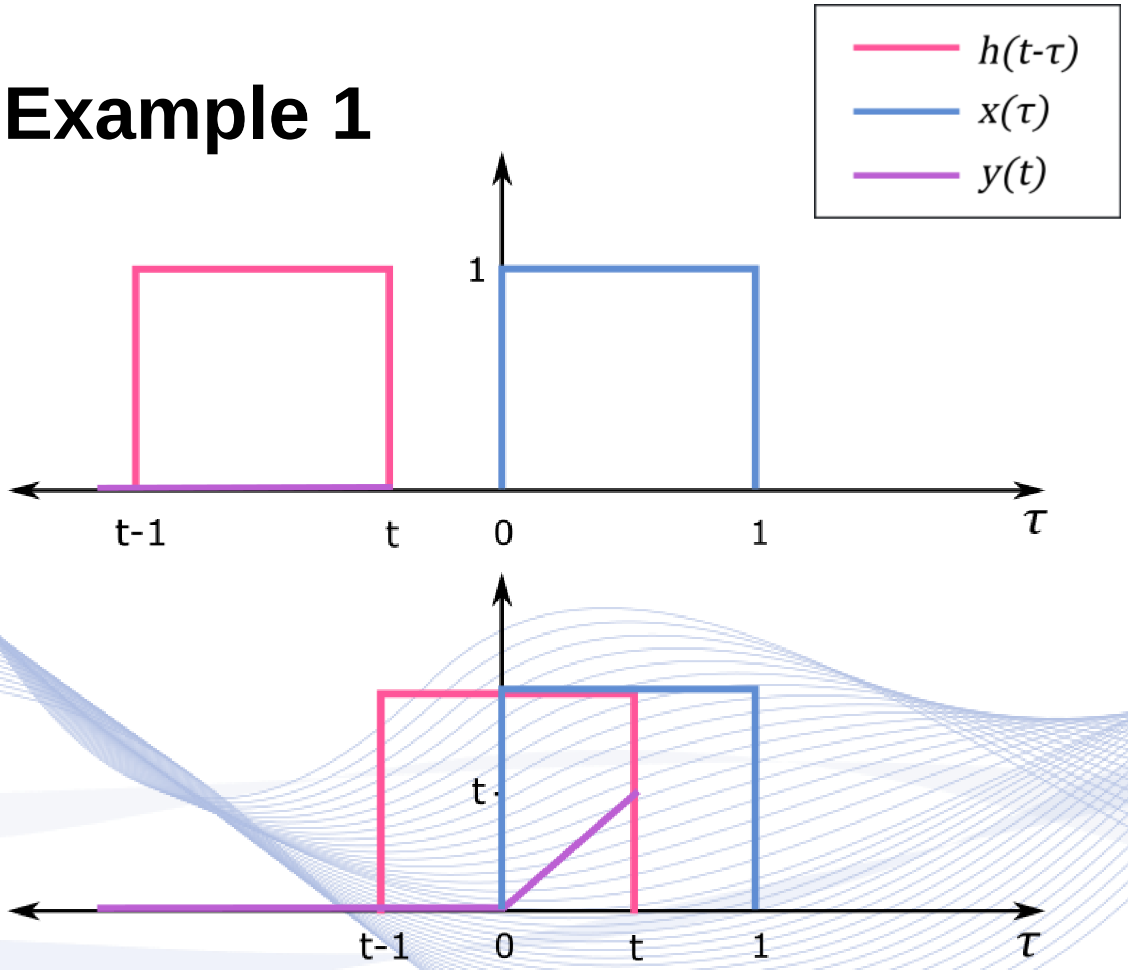
Convolution

Example 1



Convolution

Example 1



for $t < 0$:

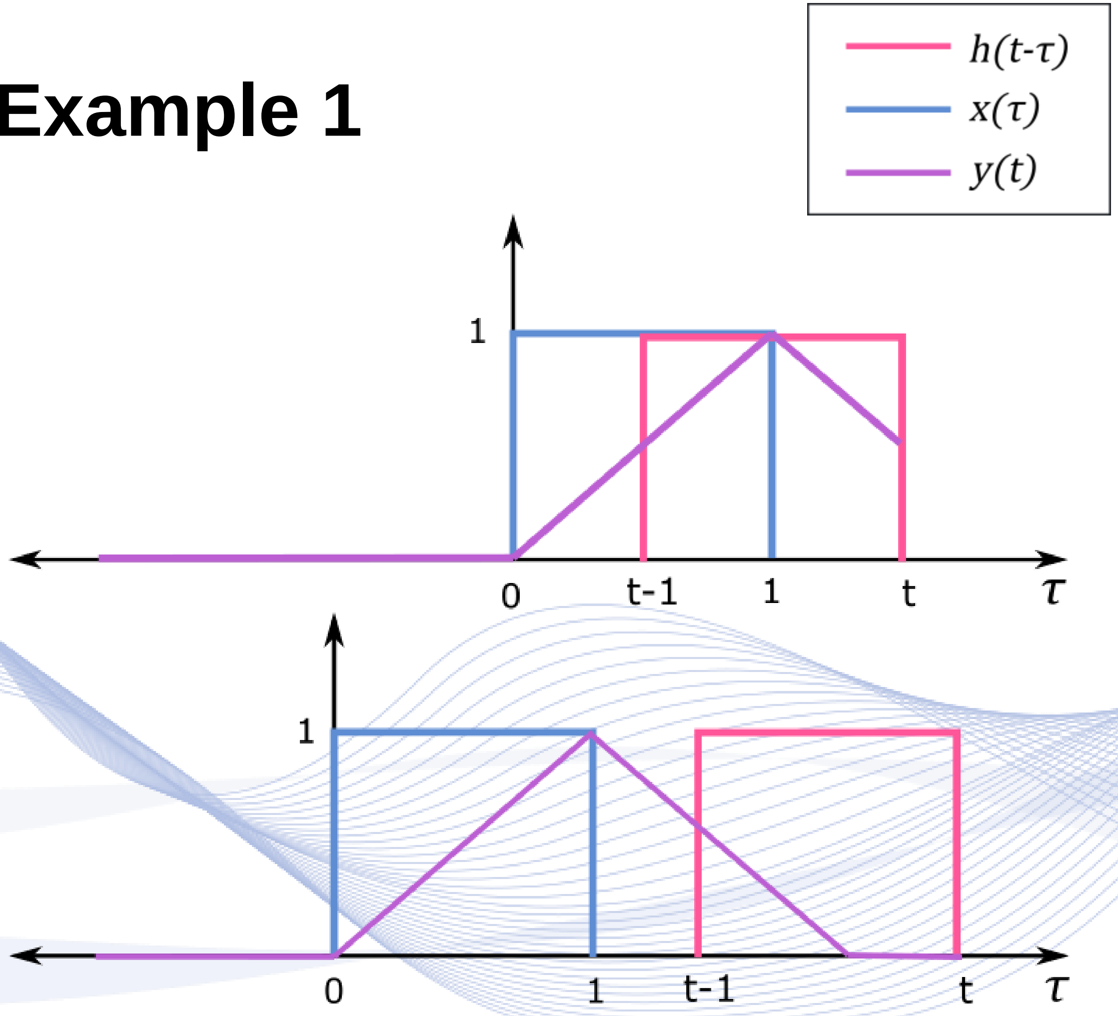
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$

for $0 \leq t < 1$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_0^t d\tau = t$$

Convolution

Example 1



for $1 \leq t < 2$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau =$$

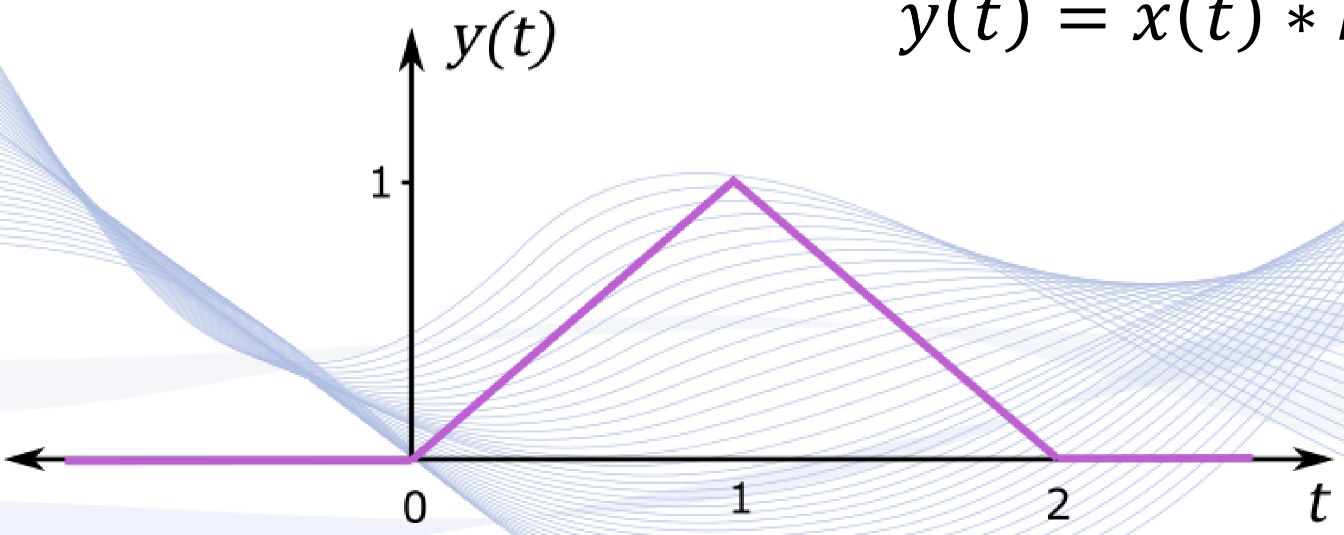
$$\int_{t-1}^{1} d\tau = 1-(t-1) = 2-t$$

for $t \geq 2$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$

Convolution

Example 1



$$y(t) = x(t) * h(t) = \begin{cases} 0, & t < 0, \\ t, & 0 \le t \le 1, \\ 2 - t, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$$

Correlation

Correlation of two signals:

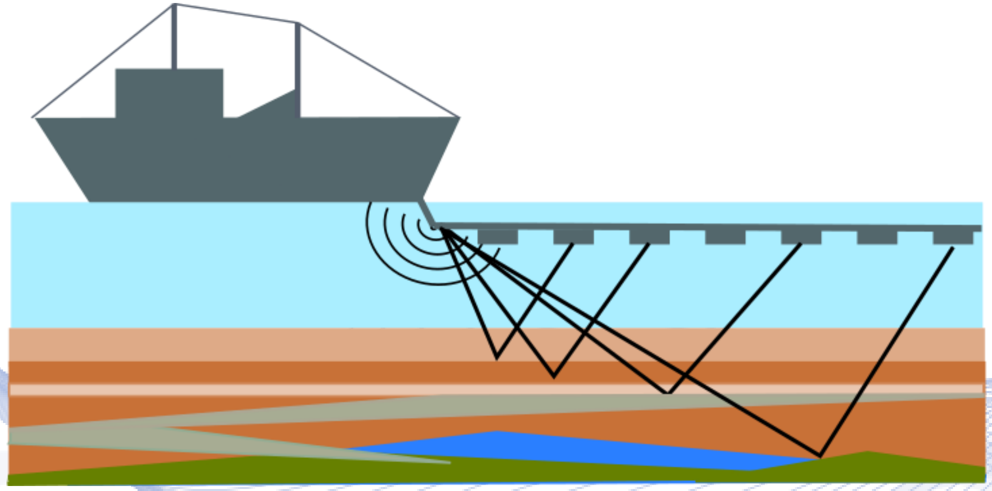
$$r_{xh}(t) = \int_{-\infty}^{\infty} x^*(\tau)h(t+\tau)d\tau.$$

- when $x(t)$ is a complex signal, the ***complex conjugate signal*** $x^*(t)$ is used in correlation.
- In correlation, unlike in convolution, $h(t)$ is not time-flipped.
- It is frequently confused with convolution. They are equivalent iff $h(t)$ is symmetric about $t = 0$.

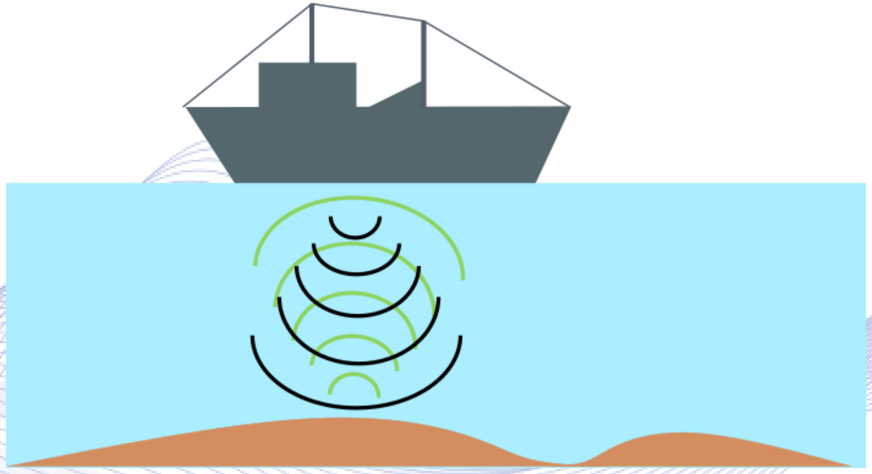
Correlation

Correlation applications:

- ***Time-of-Flight*** methods for measuring distance.



Seismic oil exploration.



Sonar.

Properties of LTI Systems

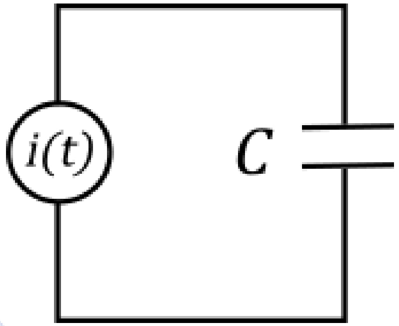
- Memory (with/without)
- Causality
- Stability (BIBO)
- Invertibility

Properties of LTI Systems

- The output of a ***system with memory*** is generated by processing both its current and the previous input values.

Examples:

- Capacitor:



$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \,.$$

- Same model can be used for integrators of any form,
  - hydraulic tanks, deposit accounts.

Properties of LTI Systems

- Unstable systems are undesirable.
- ***Stability***: A system is ***Bounded Input-Bounded Output*** (***BIBO***) ***stable***, if any bounded input results in a bounded output (i.e., output never gets infinite values).
- The impulse response of a BIBO stable system should satisfy:

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty .$$

- The absolute integrability of the impulse response $h(t)$ is a sufficient condition for BIBO stability.

Differential Equations for LTI Systems

Continuous-time LTI systems can be described by ***ordinary differential equations***:

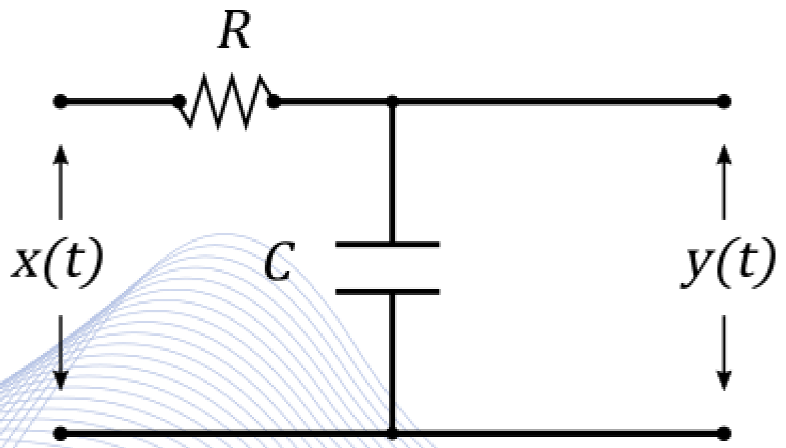
$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_0 y(t) =$$

$$b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_0 x(t).$$

Differential Equations for LTI Systems

Example:

- The input-output relation of an RC electric filter is given by:



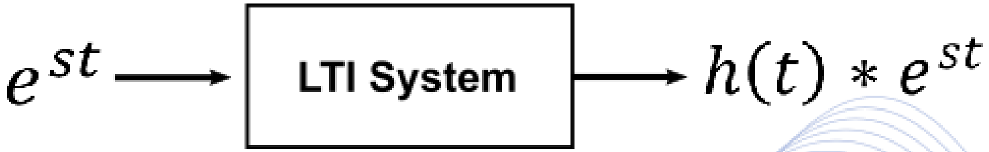
$$RC\frac{dy(t)}{dt}+y(t)=x(t).$$

- RC: RC time constant.
- It can perform delay operations.

Eigenfunctions of LTI Systems

e^{st}: ***eigenfunction*** of the LTI System.

λ: ***eigenvalue*** of the LTI system.



$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = \int_{-\infty}^{\infty} h(\tau) e^{st} e^{-s\tau} d\tau =$$

$$= [\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau]\, e^{st} = \lambda\, e^{st}$$

Q & A

Thank you very much for your attention!

More material in
http://icarus.csd.auth.gr/cvml-web-lecture-series/

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