Two-Dimensional Systems summary

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Version 3.2.1

Artificial Intelligence &
Information Analysis Lab

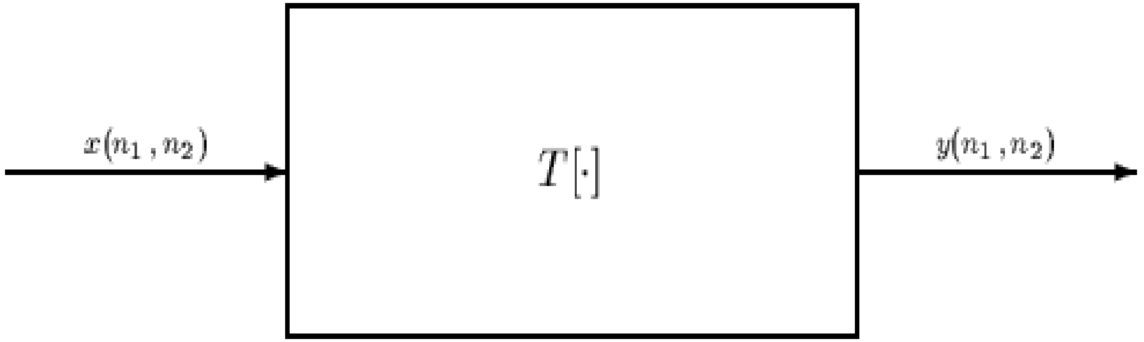
Contents

- Two-Dimensional Discrete Systems.
- Two-Dimensional $\mathcal{Z}$ Transform.
- Transfer Function of Two-Dimensional Digital Filters.
- Implementation of Two-Dimensional Digital Filters.

2D Discrete Systems

- Definition: A Two-Dimensional (2D) Discrete System T transforms a 2D signal $x(n_1, n_2)$ to a 2D output signal $y(n_1, n_2)$:

$$y(n_1, n_2) = T[x(n_1, n_2)].$$



2D Discrete System.

2D Discrete Systems

- ***Linearity***:

$$T[ax_1 + bx_2] = aT[x_1] + bT[x_2].$$

- ***Spatial Invariance***:

$$y(n_1, n_2) = T[x(n_1, n_2)] \Rightarrow$$

$$y(n_1 - m_1, n_2 - m_2) = T[x(n_1 - m_1, n_2 - m_2)].$$

2D Discrete Systems

A 2D ***Linear Spatially Invariant*** (***LSI***) system is described by a ***2D linear convolution***:

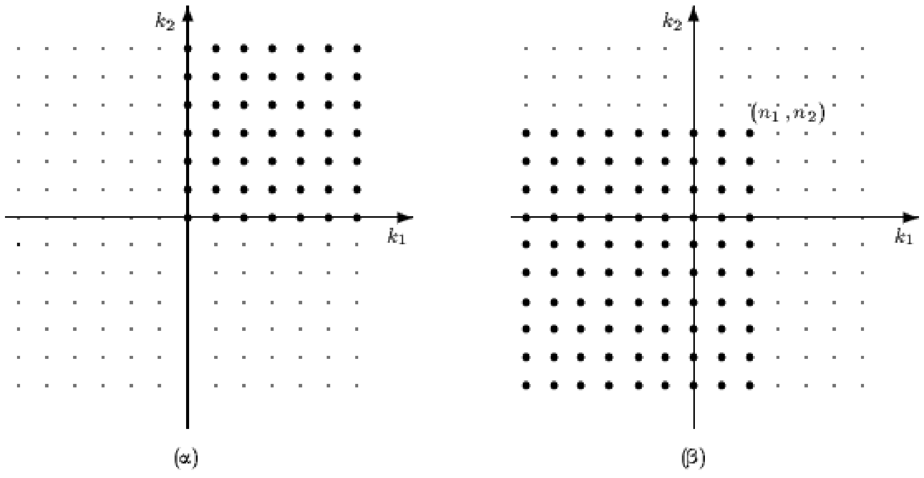
$$y(n_1,n_2) = x(n_1,n_2) ** h(n_1,n_2)$$
$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1,k_2)h(n_1-k_1,n_2-k_2).$$

An LSI system is described by its ***2D impulse response*** $h(n_1,n_2)$.

2D Discrete Systems

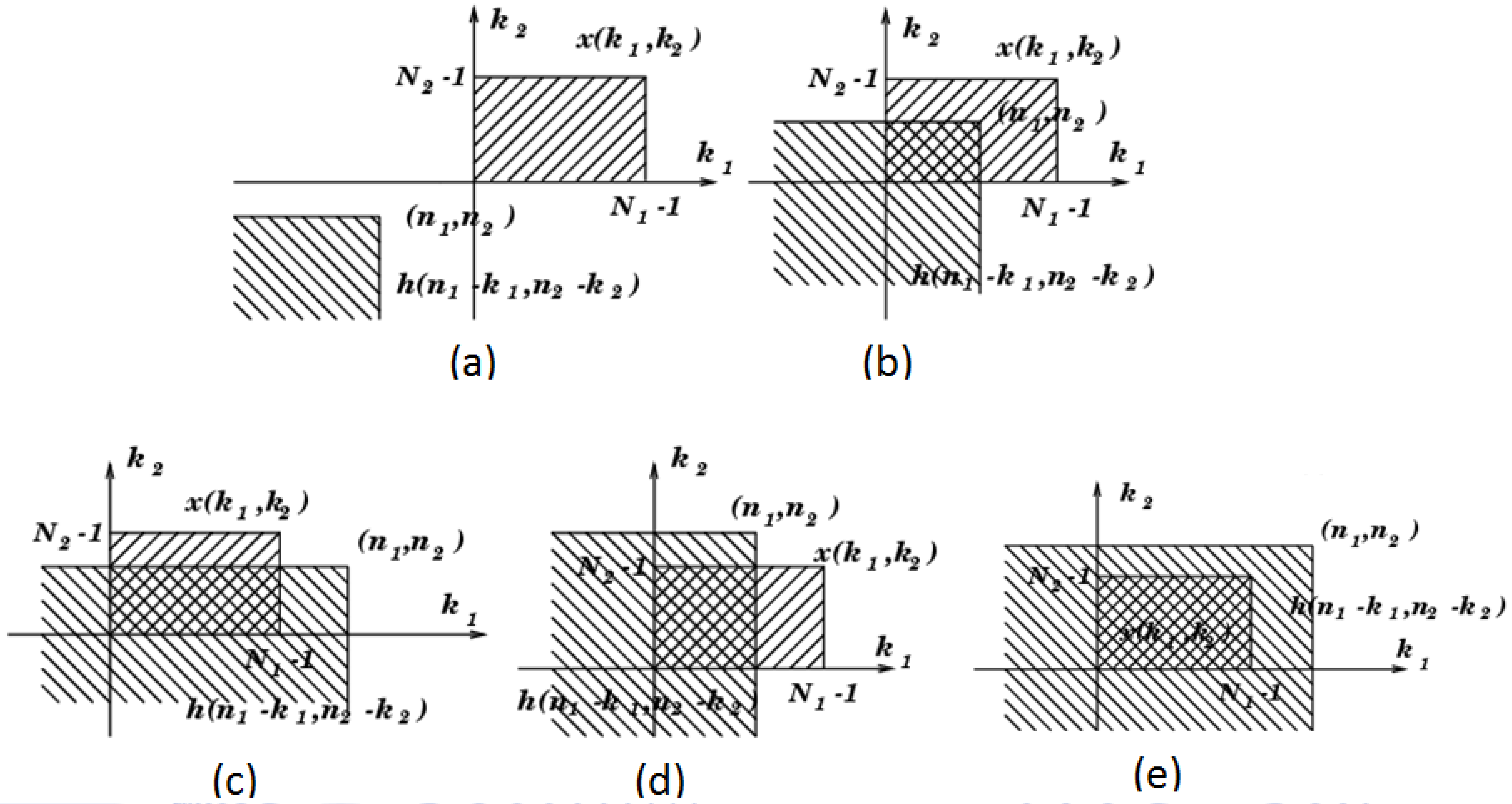
Graphical calculation of 2D convolution:

$$y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) h(n_1 - k_1, n_2 - k_2).$$



Reflection about $(0,0)$ and shift of 2D sequence $h(k_1, k_2)$: $h(n_1 - k_1, n_2 - k_2)$.

2D Discrete Systems



Visualization of 2D convolution calculation.

2D linear correlation

2D ***correlation*** of template image h and input image x (inner product):

$$r_{hx}(n_1,n_2) = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} h(k_1,k_2)\,x(k_1+n_1,k_2+n_2) = \mathbf{h}^T\mathbf{x}(n_1,n_2).$$

- $\mathbf{h} = [h(0,0),\ldots,h(N_1-1,N_2-1)]^T$: ***template image*** vector.
- $\mathbf{x}(n_1,n_2) = [x(n_1,n_2),\ldots,x(n_1+N_1-1,n_2+N_2-1)]^T$: local neighborhood (window) image vector.

2D Discrete Systems

2D system stability:

- ***Bounded Input Bounded Output*** (***BIBO***) ***stability***:

$$\forall(n_1,n_2) \quad |x(n_1,n_2)| \le B \quad \rightarrow$$
$$\exists B': \quad |y(n_1,n_2)| \le B'.$$

- Sufficient and necessary condition of for BIBO stability:

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |h(n_1,n_2)| = S_1 < \infty .$$

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2D Discrete Systems

The impulse response support distinguishes 2D systems in:

- ***2D Finite Impulse Response*** (***FIR***) systems have a finite filter window of size $M_1 \times M_2$ samples: $0 \leq n_1 < M_1, \quad 0 \leq n_2 < M_2$.

They are described by the 2D convolution:

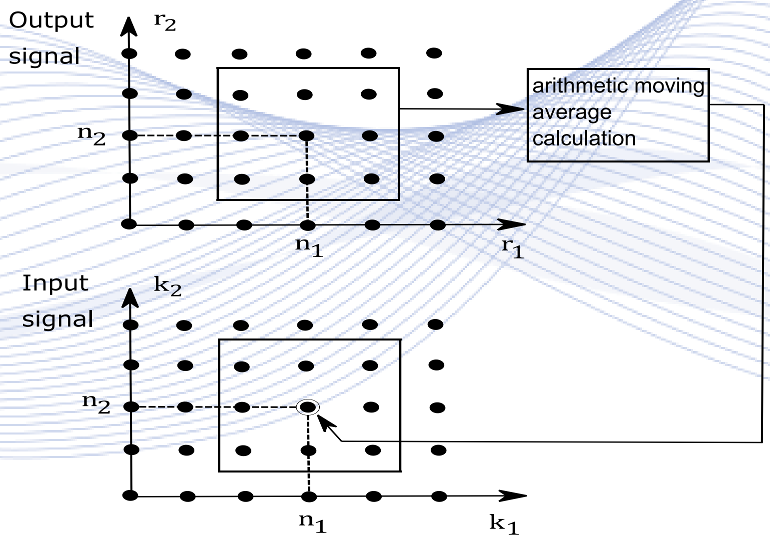
$$y(n_1, n_2) = \sum_{k_1=0}^{M_1-1} \sum_{k_2=0}^{M_2-1} h(k_1, k_2) x(n_1 - k_1, n_2 - k_2).$$

2D Discrete Systems

Example of an FIR is the arithmetic moving average filter:

$$y(n_1, n_2) = \frac{1}{M_1 M_2} \sum_{k_1=-v_1}^{v_1} \sum_{k_2=-v_2}^{v_2} x(n_1 - k_1, n_2 - k_2).$$

- Odd window size: $M_i = 2v_i + 1, \quad i = 1,2.$



2D Discrete Systems

Moving average image filtering.

2D Discrete Systems

- ***2D Infinite Impulse Response*** (***IIR***) systems:

  - They do not have a space-limited impulse response support.
  - They are described by the difference equation:

$$\sum_{k_1}\sum_{k_2} b(k_1,k_2)y(n_1-k_1,n_2-k_2) =$$

$$\sum_{r_1}\sum_{r_2} a(r_1,r_2)x(n_1-r_1,n_2-r_2).$$

2D Discrete Systems

2D IIR filter (edge detector) output.

2D $\mathcal{Z}$ Transform

- Definition of ***2D $\mathcal{Z}$ Transform***:

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) z_1^{-n_1} z_2^{-n_2}$$

- It performs a mapping $\mathbb{Z}^2 \longrightarrow \mathbb{C}$.
- It can be considered as a 2-variable polynomial of z_1^{-1}, z_2^{-1}.

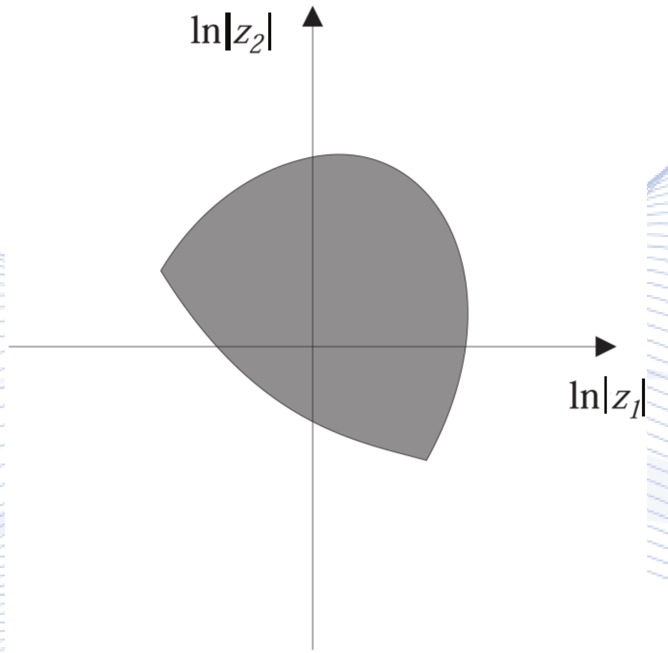
- Definition of ***inverse 2D $\mathcal{Z}$ Transform***:

$$x(n_1, n_2) = \left(\frac{1}{2\pi i}\right)^2 \oint_{C_1} \oint_{C_2} X(z_1, z_2) z_1^{n_1-1} z_2^{n_2-1} dz_1 dz_2 .$$

2D $\mathcal{Z}$ Transform

- ***Region of Convergence*** (***ROC***) is region of the 2D plane $(|z_1|, |z_2|)$ where 2D $\mathcal{Z}$ Transform converges:

$$\sum_{n_1}\sum_{n_2}|x(n_1,n_2)|\,|z_1|^{-n_1}|z_2|^{-n_2} = S_1 < \infty.$$



Region of Convergence of 2D $\mathcal{Z}$ Transform.

Transfer Function of 2D Digital Filters

- ***2D transfer function*** definition:

$$H(z_1, z_2) = \frac{Y(z_1, z_2)}{X(z_1, z_2)}.$$

- Transfer function of a 2D FIR filter:

$$H(z_1, z_2) = \sum_{k_1=0}^{M_1-1} \sum_{k_2=0}^{M_2-1} h(k_1, k_2) z_1^{-k_1} z_2^{-k_2}.$$

- It is a 2-variable polynomial of z_1^{-1}, z_2^{-1}.
- Such polynomials: a) can not be easily factorized; b) they do not have distinct roots.

Transfer Function of 2D Digital Filters

- Transfer function of a 2D IIR filter:

$$H(z_1, z_2) = \frac{\sum_{r_1} \sum_{r_2} a(r_1, r_2) z_1^{-r_1} z_2^{-r_2}}{\sum_{k_1} \sum_{k_2} b(k_1, k_2) z_1^{-k_1} z_2^{-k_2}} = \frac{A(z_1, z_2)}{B(z_1, z_2)}.$$

- It is a 2D rational function of z_1^{-1}, z_2^{-1}.
- Denominator polynomial $B(z_1, z_2)$ may become 0, leading the IIR system to instability. It cannot be easily factorized.

Stability of 2D Digital Filters

- An easy solution to IIR stability is to employ separable-denominator IIR filters of the form:

$$H(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)} = \frac{A(z_1, z_2)}{B_1(z_1)B_2(z_2)}.$$

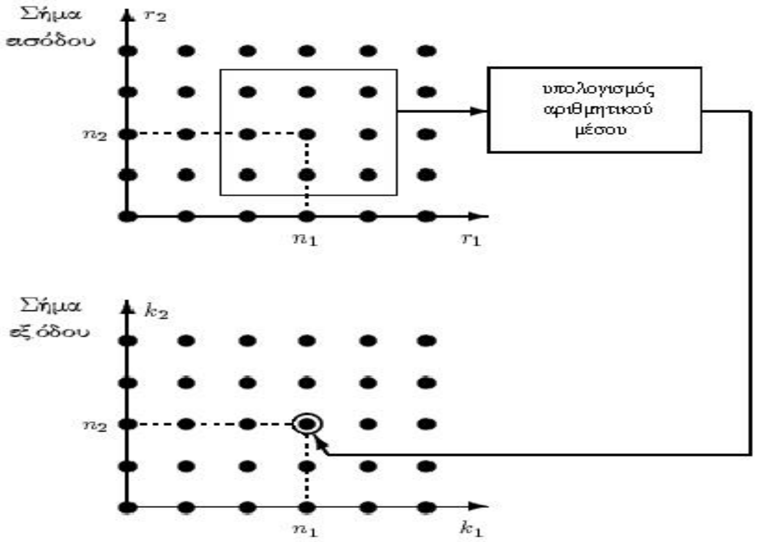
- The zeros of polynomials $B_1(z_1), B_2(z_2)$ are called system ***poles***.
- If poles are inside the respective unit circles $|z_1| < 1, |z_2| < 1$, such an IIR system is stable.

Implementation of 2D Digital Filters

Implementation of 2D FIR filters:

$$y(n_1, n_2) = \sum_{k_1=0}^{M_1-1} \sum_{k_2=0}^{M_2-1} h(k_1, k_2)\, x(k_1 - n_1, k_2 - n_2).$$

- Image scan be scanned row-wise or column-wise or using any other scanning scheme (e.g., zig-zag).



Implementation of 2D Digital Filters

- FIR output calculation at each scanning location can be parallelized.

- It is ideal for SIMD computers, e.g., GPU computing.

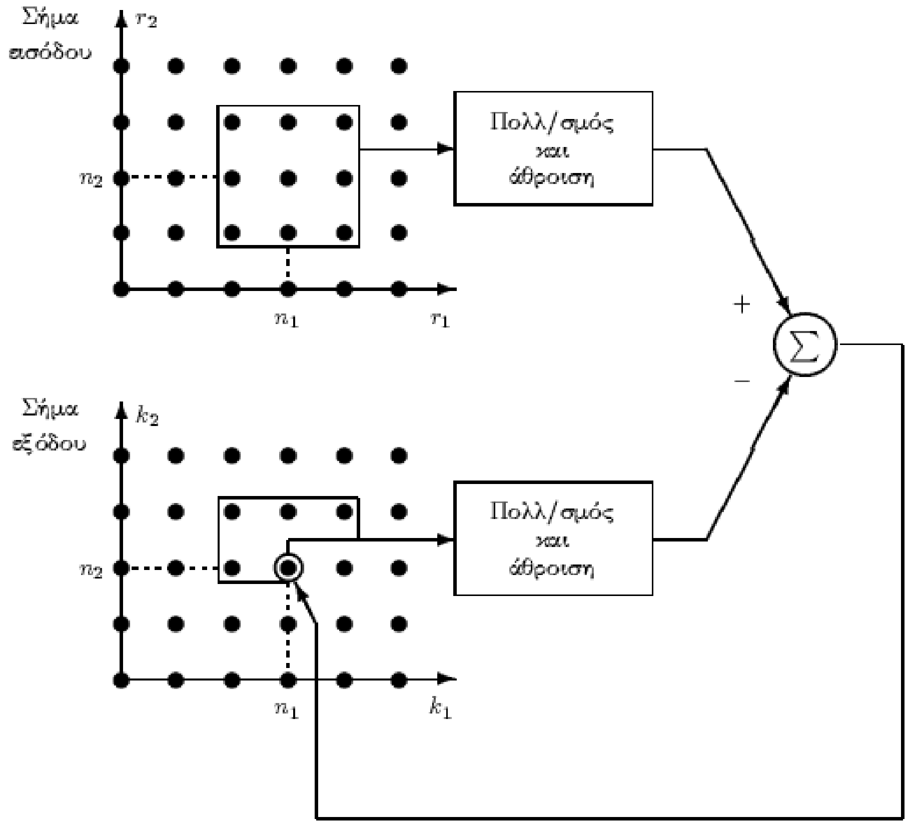
Implementation of 2D Digital Filters

Implementation of 2D IIR filters:

$$y(n_1,n_2) =$$
$$= \sum_{r_1}\sum_{r_2} a(r_1,r_2)x(n_1-r_1,n_2-r_2) - \sum_{k_1}\sum_{k_2} b(k_1,k_2)y(n_1-k_1,n_2-k_2),$$

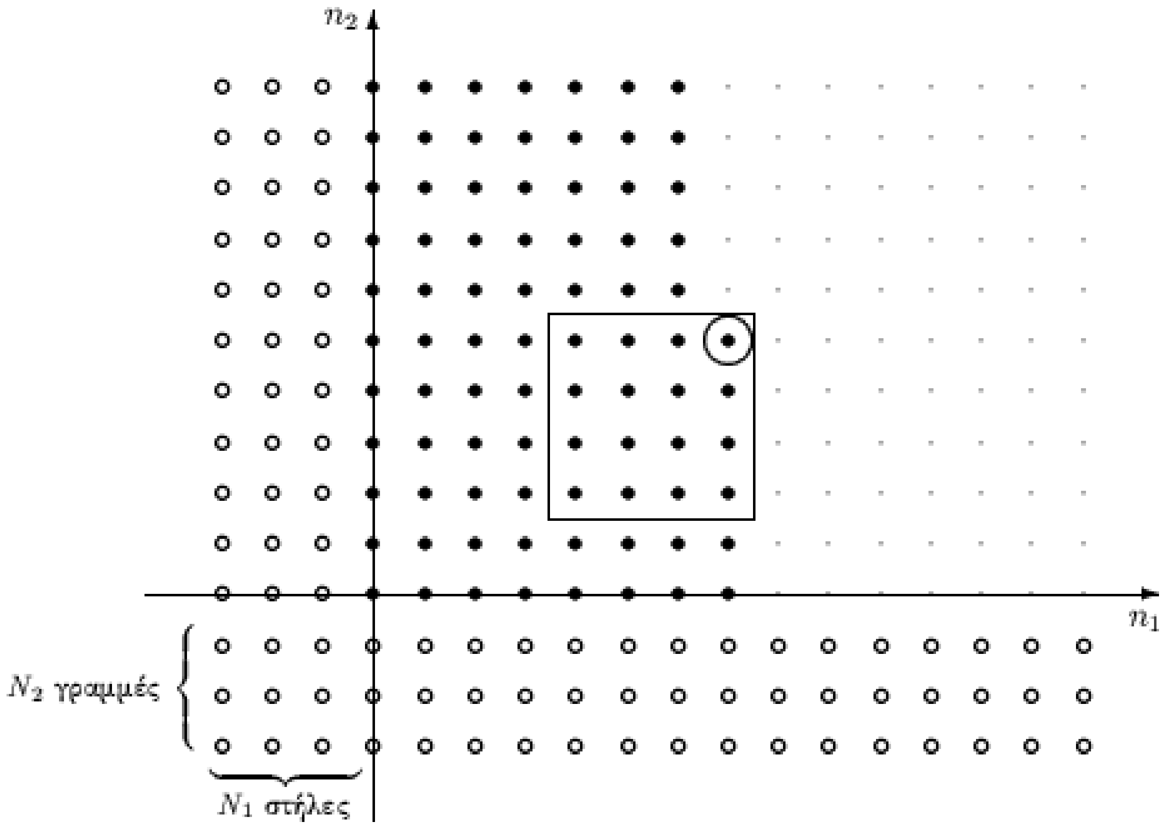
where $(k_1,k_2) \neq (0,0)$.

Implementation of 2D Digital Filters



2D IIR filter implementation.

Implementation of 2D Digital Filters



Output calculation of a first quadrant IIR filter.

Implementation of 2D Digital Filters

- Despite stability and output mask shape issues, IIR filters may have much less computations than FIR filters of similar characteristics.
- Their serial execution can be very fast.
- However, their parallelization is not obvious.

- Hence, their execution in SIMD machines (e.g., on GPU cards) must be considered with care.

Bibliography

[PIT2021] I. Pitas, “Computer vision”, Createspace/Amazon, in press.
[PIT2017] I. Pitas, “Digital video processing and analysis” , China Machine Press, 2017 (in Chinese).
[PIT2013] I. Pitas, “Digital Video and Television” , Createspace/Amazon, 2013.
[NIK2000] N. Nikolaidis and I. Pitas, “3D Image Processing Algorithms”, J. Wiley, 2000.
[PIT2000] I. Pitas, “Digital Image Processing Algorithms and Applications”, J. Wiley, 2000.

Q & A

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