Set Theory summary

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Set theory

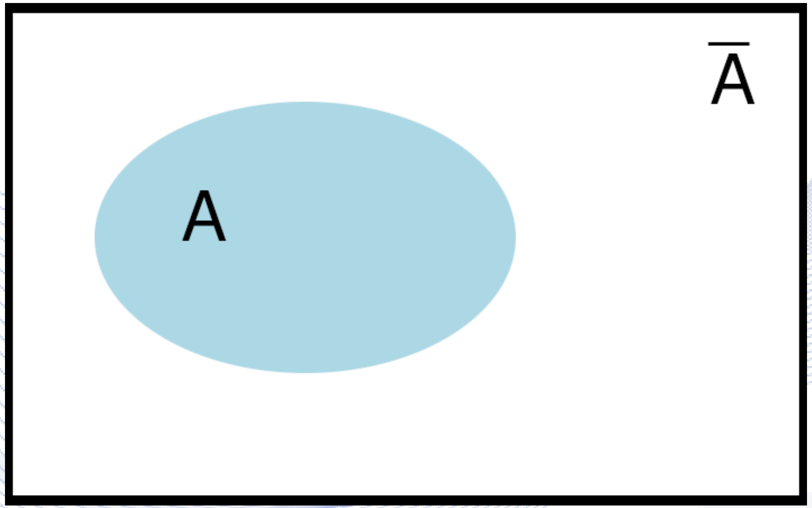
- **Sets**
- Fuzzy sets
- Applications

Sets

- A set of n set members $z_i = 1,..,n$ is denoted by:

$$\mathcal{A} = \{z_1, \ldots, z_n\}.$$

- $|\mathcal{A}| = n$: ***set cardinality***.



- If $\mathcal{B}$ is a ***subset*** of $\mathcal{A}$, it is denoted as :

$$\mathcal{B} \subset \mathcal{A} \equiv z \in \mathcal{B} \Rightarrow z \in \mathcal{A}.$$

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Sets

- Subsets of $\mathcal{A}$ are the null set $\emptyset$, as well as sets containing any number of set $\mathcal{A}$ members z_i.
- Always:

$$\{\emptyset\} \subset \mathcal{A}, \qquad \mathcal{A} \subseteq \mathcal{A}.$$

- If $\mathcal{E}$ is the ***universal set*** of all sets :

$$\mathcal{A} \subset \mathcal{E}, \quad \forall \mathcal{A}.$$

- ***Power set*** of a set $\mathcal{A}$ has members all possible subsets of $\mathcal{A}$.

Sets

- ***Set union*** $\mathcal{A} \cup \mathcal{B}$ is defined by:

$$z \in \mathcal{A} \text{ or } z \in \mathcal{B} \Rightarrow z \in (\mathcal{A} \cup \mathcal{B}).$$

- ***Set intersection*** $\mathcal{A} \cap \mathcal{B}$ is defined by:

$$z \in \mathcal{A} \text{ and } z \in \mathcal{B} \Rightarrow z \in (\mathcal{A} \cap \mathcal{B}).$$

Sets

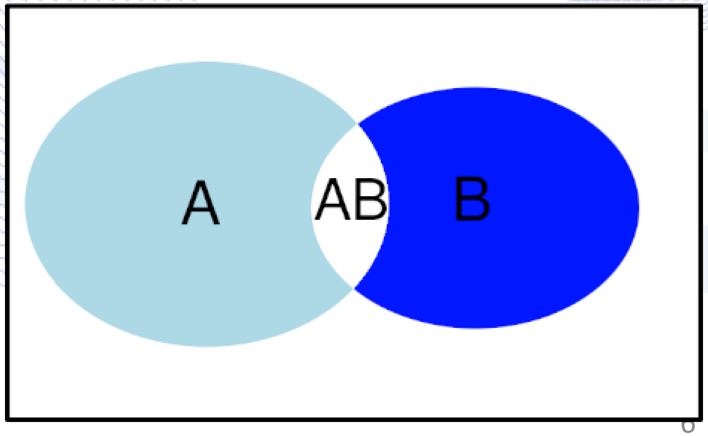
- Set operation properties:

$$\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A},$$

$$(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C}) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C},$$

$$\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A},$$

$$(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C}) = \mathcal{A} \cap \mathcal{B} \cap \mathcal{C}.$$



Sets

- ***Borel Field*** (σ-field) is a field having the following additional property:

$$\mathcal{A}_i \in \mathcal{F} \Rightarrow (\cup_i \mathcal{A}_i) \in \mathcal{F} \Rightarrow (\cap_i \mathcal{A}_i) \in \mathcal{F}$$

for any possible countable infinite sequence $\mathcal{A}_i$.

- It can be proven that the collection of all $\mathcal{F}$ subsets of the set $\mathcal{E}$ is a Borel Field.

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Set theory

- Sets
- **Fuzzy sets**
- Applications

Fuzzy sets

- Let $\mathcal{E}$ is a universal set of objects $z \in \mathcal{E}$.
- Crisp set $\mathcal{A}$ ($\mathcal{A} \subset \mathcal{E}$) membership function $A_i(z)\colon \mathcal{E} \rightarrow \{0,1\}$:

$$A_i(z) = \begin{cases} 1, & \text{if } z \in \mathcal{A} \\ 0, & \text{if } z \notin \mathcal{A}. \end{cases}$$

- ***Fuzzy set*** $\mathcal{A}$ ($\mathcal{A} \subset \mathcal{E}$) ***membership function***:

$$A_i(z)\colon \mathcal{E} \rightarrow [0,1].$$

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Fuzzy sets

- Fuzzy set $\mathcal{A}$ is fully characterized as follows :

$$\mathcal{A} = \{(z, A_i(z)), z \in \mathcal{E}\}.$$

- When $\mathcal{E} = \{z_1, \ldots, z_n\}$ is a finite element set, a fuzzy set $\mathcal{A}$ is expressed by:

$$\mathcal{A} = A_i(z_1)/z_1 + \cdots + A_i(z_n)/z_n.$$

Set theory

- Sets
- Fuzzy sets
- **Applications**

Images as sets

- An image domain is set:
- Continuous images: $\mathcal{X} \in \mathbb{R}^2$.
- digital images (set of *pixels*): $\mathcal{X} \in \mathbb{Z}^2$.
- Typically is $\mathcal{X} \in \mathbb{Z}^2$ is a $N \times M$ rectangle of pixels.

Volumetric images:

- Their domain (set of *voxels*): $\mathcal{X} \in \mathbb{Z}^3$.

Image domain $\mathcal{X}$.

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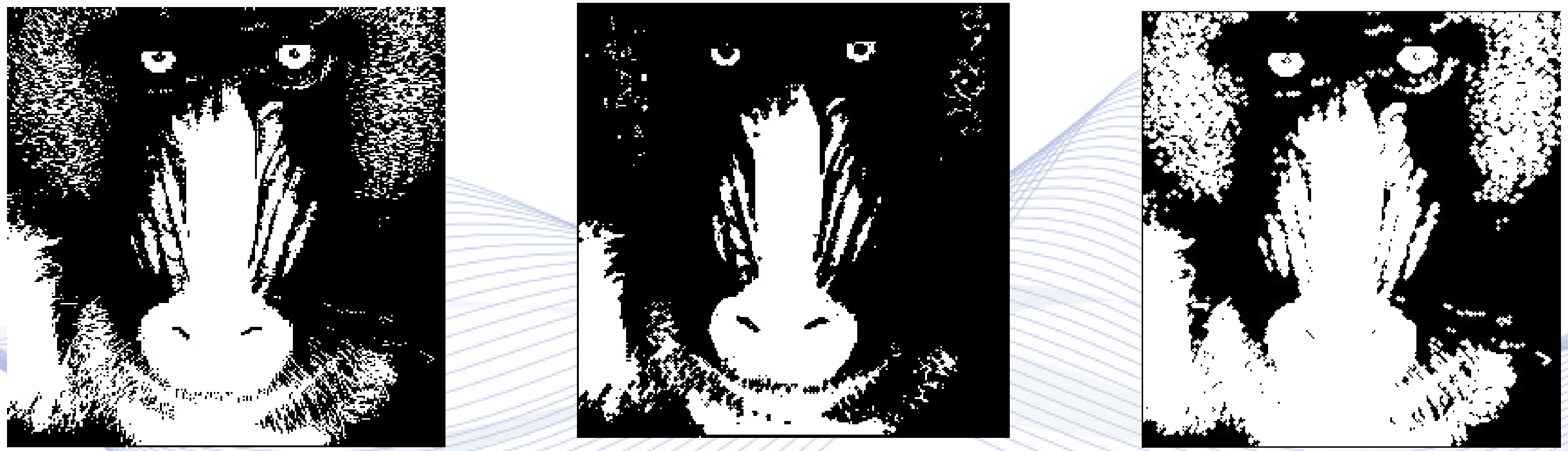
Mathematical morphology

Erosion and dilation are special cases of ***Minkowski set addition*** and ***Minkowski set subtraction***:

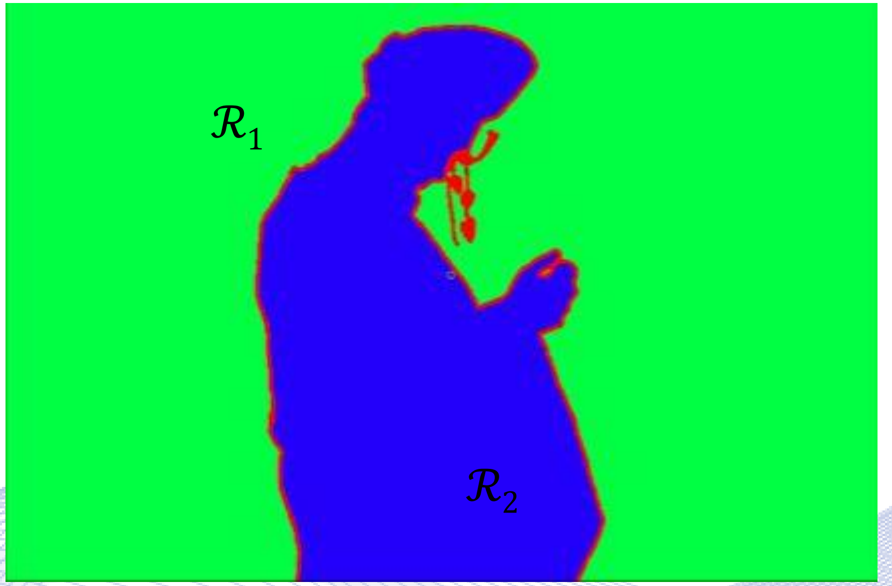
$$\mathcal{X} \oplus \mathcal{B}^s = \bigcup_{b \in \mathcal{B}} \mathcal{X}_b, \qquad \mathcal{X} \ominus \mathcal{B}^s = \bigcap_{b \in \mathcal{B}} \mathcal{X}_b.$$



(a) (b) (c)

a) Thresholded image; b) eroded and c) dilated image by the structuring elements SQUARE.

Image segmentation



Foreground and background segmentation.

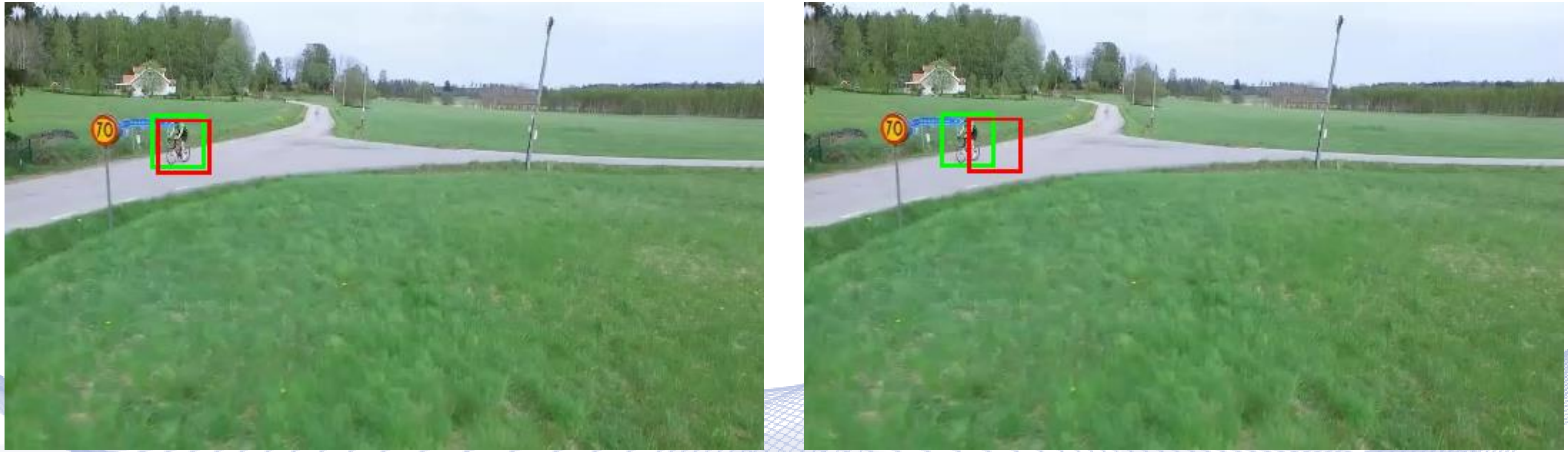
Object Detection Performance Metrics

Intersection over Union (IoU):

$$J(\mathcal{A},\mathcal{B}) = |\mathcal{A} \cap \mathcal{B}|/|\mathcal{A} \cup \mathcal{B}|.$$

- $\mathcal{A},\mathcal{B}$: estimated, ground truth ROIs (sets, bounding boxes).
- $|\mathcal{A}|$: set cardinality (area counted in pixels)
- Also called ***Jaccard Similarity Coefficient*** or ***Overlap Score.***

Object Detection Performance Metrics

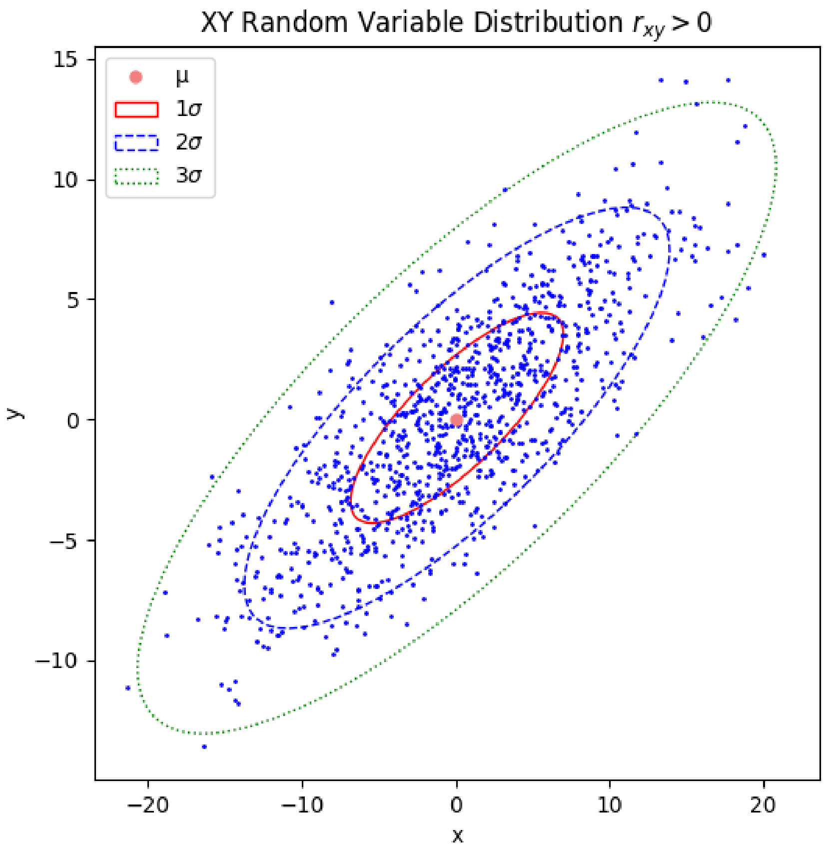


Object detection: a) $J(\mathcal{A},\mathcal{B}) = 0.67$; b) $J(\mathcal{A},\mathcal{B}) = 0.27$.

Data sets

- An object can be represented by a real-valued feature vectors: $\mathbf{x} \in \mathbb{R}^n$.

- A data set is a feature vector set: $\mathcal{D} = \{\mathbf{x}_i\}_{i=1}^N$.



Data set $\mathcal{D}$.

Crisp/fuzzy set clustering

Let $\mathcal{D}$ be a feature data set: $\mathcal{D} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$.

- ***Crisp clustering*** is the partition of $\mathcal{D}$ into m sets $\mathcal{C}_1, \dots, \mathcal{C}_m$, satisfying the following conditions:
  - $\mathcal{C}_i \neq \emptyset, \quad i = 1, \dots, m,$
  - $\cup_{i=1}^{m} \mathcal{C}_i = \mathcal{D},$
  - $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset, \quad i \neq j, \quad i, j = 1, \dots, m.$
- Feature vectors $\mathbf{x}_i$ in a cluster $\mathcal{C}_i$ are 'similar', while they are 'dissimilar' to the ones of other clusters $\mathcal{C}_j, i \neq j$.