Geometry summary

Dr. I. Mademlis, Prof. Ioannis Pitas
Aristotle University of Thessaloniki
pitas@csd.auth.gr
www.aiia.csd.auth.gr
Version 2.6

Artificial Intelligence &
Information Analysis Lab

Geometry

- **Vector calculus**
- 3D geometric transformations
- Projective geometry

Vector calculus

- Vectors of the form $\mathbf{P} = [X, Y, Z]^T$ define point positions in the 3D space $\mathbb{R}^3$.

- Vectors of the form $\mathbf{p} = [x, y]^T$ define point positions in the 2D space $\mathbb{R}^2$.

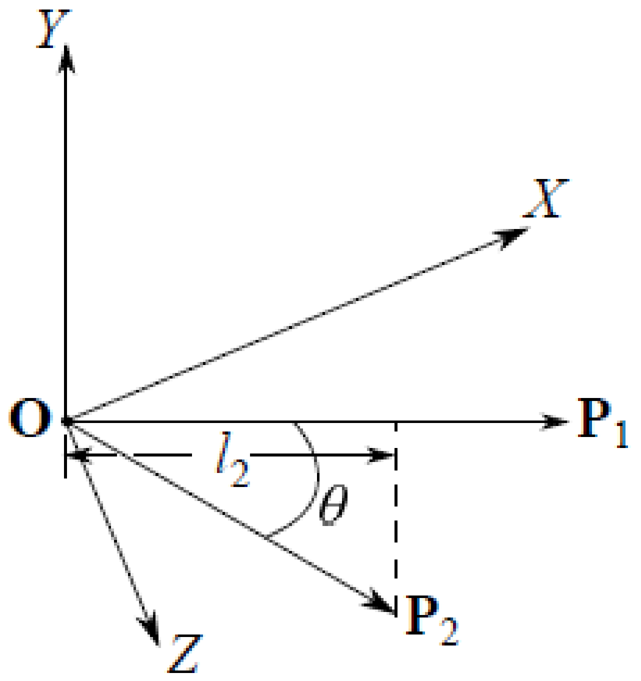
- The right-hand thumb rule is typically followed when defining the axis system (X, Y, Z) in $\mathbb{R}^3$.

Vector calculus

Inner vector product or ***dot product*** in $\mathbb{R}^3$:

$$\mathbf{P}_1^T\mathbf{P}_2 = \mathbf{P}_1\mathbf{P}_2^T = X_1X_2 + Y_1Y_2 + Z_1Z_2$$
$$= \|\mathbf{P}_1\| \cdot \|\mathbf{P}_2\| \cos\theta \,.$$

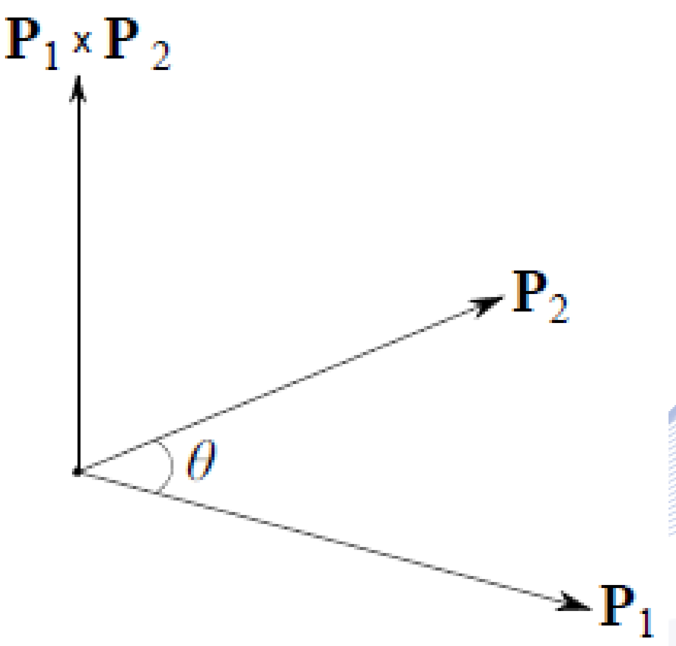
- θ: the angle formed by the two vectors.



Vector calculus

Cross vector product $\mathbf{P}_1 \times \mathbf{P}_2$ in $\mathbb{R}^3$:

$$\mathbf{P}_1 \times \mathbf{P}_2 = (Y_1 Z_2 - Z_1 Y_2)\mathbf{i} + (Z_1 X_2 - X_1 Z_2)\mathbf{j} + (X_1 Y_2 - Y_1 X_2)\mathbf{k}.$$



- $\mathbf{i}, \mathbf{j}, \mathbf{k}$: the standard basis vectors of $\mathbb{R}^3$.
- It is a vector perpendicular to plane defined by vectors $\mathbf{P}_1, \mathbf{P}_2$.

Geometry

- Vector calculus
- **3D geometric transformations**
- Projective geometry

3D geometric transformations

- ***3D solid object*** motions is superposition of a 3D rotation and a 3D translation:

$$\mathbf{X}' = \mathbf{R}\mathbf{X} + \mathbf{T}$$

- $\mathbf{X} = [X, Y, Z]^T$, $\mathbf{X}' = [X', Y', Z']^T$: the coordinates of a solid object point at time instances t and t'.
- $\mathbf{T} = [T_X, T_Y, T_Z]^T$: a 3D translation vector.
- Rotation can precede translation, or vice versa:

$$\mathbf{X}' = \mathbf{R}(\mathbf{X} + \mathbf{T}).$$

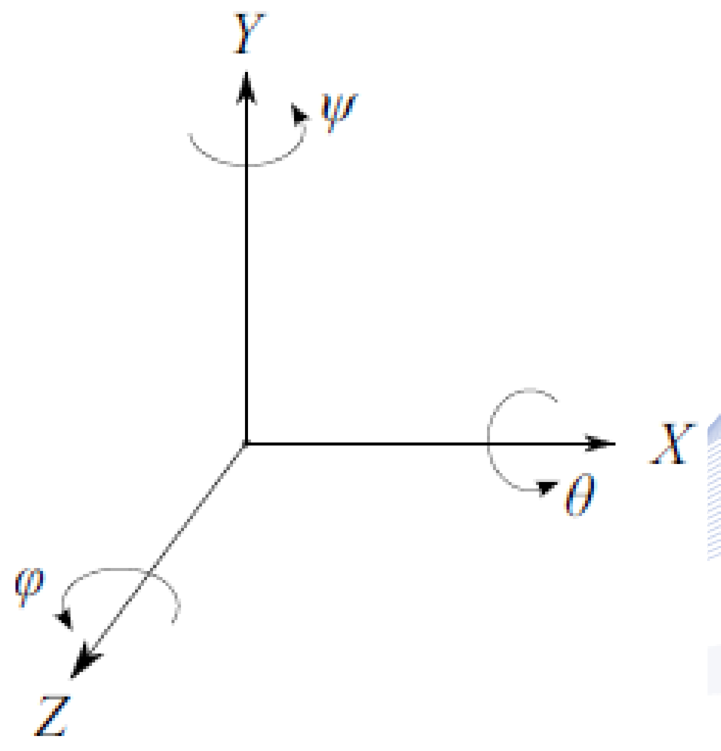
3D geometric transformations

- $\mathbf{R}$ is a 3×3 rotation matrix, which can be defined by either:
  - The Euler rotation angles about X, Y, Z axes (in Cartesian coordinates)
  - a unitary rotation axis and a rotation angle about this axis.
- $\mathbf{T} = [T_X, T_Y, T_Z]^T$: a 3D translation vector.

3D geometric transformations

- An arbitrary rotation in the 3D space can be represented by the Euler rotation angles θ, ψ, ϕ about the X, Y, Z axes.
- Each can be described by a 1D rotation matrix, leading to:

$$\mathbf{R} = \mathbf{R}_Z \mathbf{R}_Y \mathbf{R}_X.$$



3D geometric transformations

- 3D rotation can also be represented by ***quaternions*** that are extensions of complex numbers:

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}.$$

- q_0, q_1, q_2, q_3 are real numbers and:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

- Unit quaternion $\mathbf{q}_R = [q_0\ q_1\ q_2\ q_3]^T$ satisfies:

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1.$$

3D geometric transformations

- Rotation by an angle α around a unit vector $[n_1, n_2, n_3]^T$:

$$\mathbf{q} = \left[n_1 \sin\frac{\alpha}{2} \quad n_2 \sin\frac{\alpha}{2} \quad n_3 \sin\frac{\alpha}{2} \quad \cos\frac{\alpha}{2} \right]^T.$$

- Rotation matrix $\mathbf{B}$ corresponding to a certain quaternion:

$$\mathbf{R} = \begin{bmatrix} q_0^2 - q_1^2 - q_2^2 - q_3^2 & 2(q_0q_1 + q_2q_3) & 2(q_0q_2 - q_1q_3) \\ 2(q_0q_1 - q_2q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1q_2 + q_0q_3) \\ 2(q_0q_2 + q_1q_3) & 2(q_1q_2 - q_0q_3) & -q_0^2 + q_1^2 + q_2^2 + q_3^2 \end{bmatrix}.$$

Geometry

- Vector calculus
- 3D geometric transformations
- **Projective geometry**

Projective geometry

- *Homogeneous coordinates* are basic concept in projective geometry.
- Assuming an Euclidean plane $\mathbb{R}^2$ and a Cartesian coordinate system defined on it, $\mathbf{p} = [x, y]^T$, we assign the ordered 3-tuple $[x_h, y_h, a]^T$ to $\mathbf{p}$, where $a \in \mathbb{R}, a \neq 0, x = x_h/a\,, y = y_h/a$.
- The homogeneous coordinates $[x_h, y_h, a]^T$ define the projective plane $\mathbb{P}^2$.
    - Exactly one Euclidean point corresponds to each 3-tuple $[x_h, y_h, a]^T$.
    - But both $[x_h, y_h, a]^T$ and $[\lambda x_h, \lambda y_h, \lambda a]^T$ 3-tuples correspond to the same Euclidean coordinates $[x, y]^T$, where $\lambda \in \mathbb{R}, \lambda \neq 0$.
    - Canonical form $[x_h, y_h, 1]^T$.
    - The projective plane can be defined over the real or the complex field $x_h, y_h, a \in \mathbb{C}$.

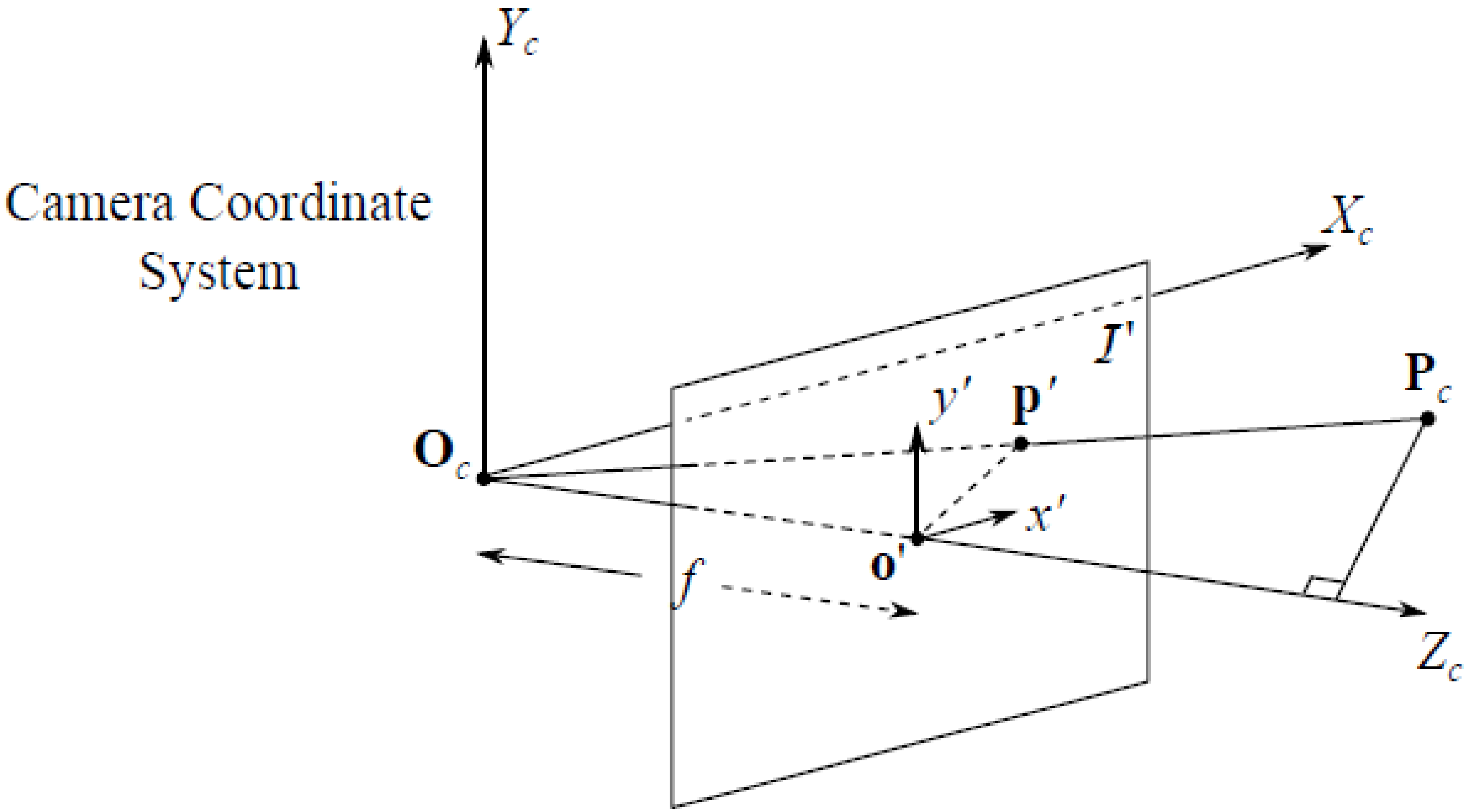
Projective geometry

- Assuming a 3D Euclidean space $\mathbb{R}^3$ and a Cartesian coordinate system defined on it, $\mathbf{P} = [X, Y, Z]^T$, we assign the ordered 4-tuple $[X_h, Y_h, Z_h, a]^T$ to $\mathbf{P}$, where $a \in \mathbb{R}, a \neq 0, X = X_h/a\,, Y = Y_h/a\,, Z = Z_h/a$.

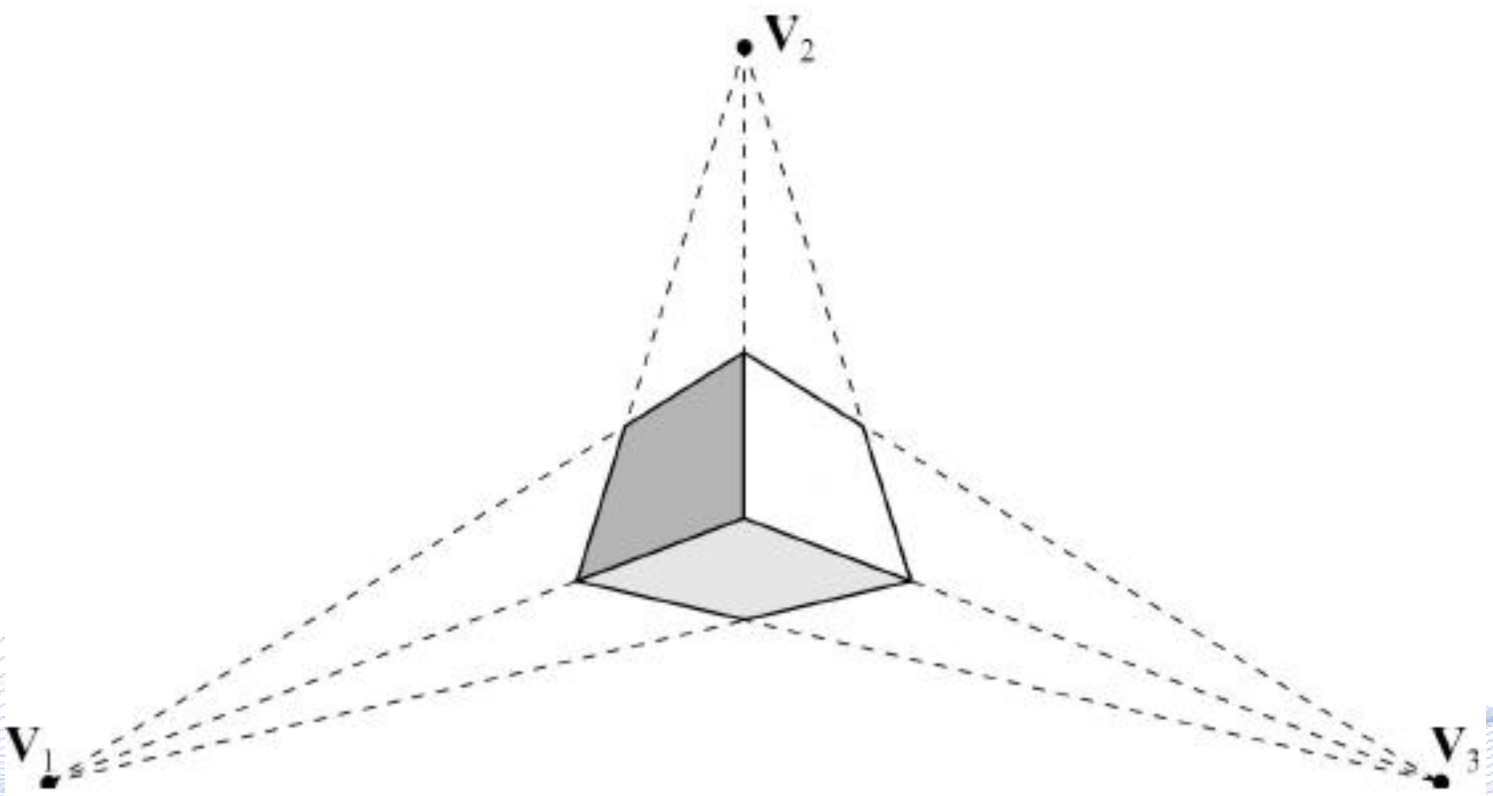
- The homogeneous coordinates $[X_h, Y_h, Z_h, a]^T$ define the projective space $\mathbb{P}^3$.
  - Exactly one Euclidean point corresponds to each 4-tuple $[X_h, Y_h, Z_h, a]^T$.
  - But both 4-tuples $[X_h, Y_h, Z_h, a]^T$ and $[\lambda X_h, \lambda Y_h, \lambda Z_h, \lambda a]^T$ correspond to the same 3D Euclidean coordinates $[X, Y, Z]^T$, where $\lambda \in \mathbb{R}, \lambda \neq 0$.
  - $[X_h, Y_h, Z_h, 1]^T$.

Projective geometry

Pinhole camera: projection of the 3D world $\mathbb{R}^3$ on an image plane $\mathbb{R}^2$.

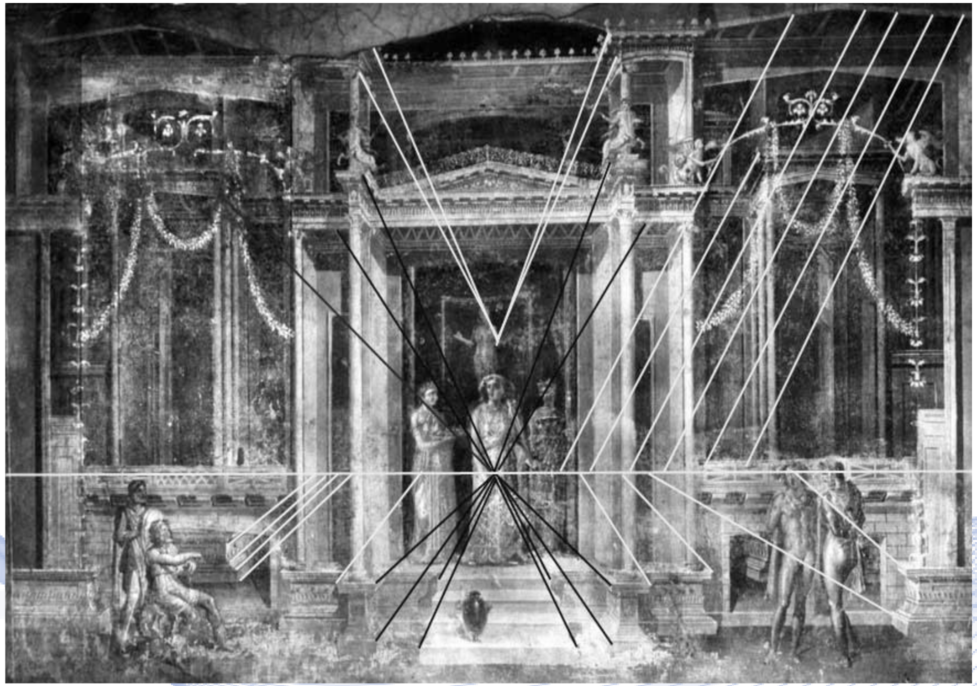


Projective geometry



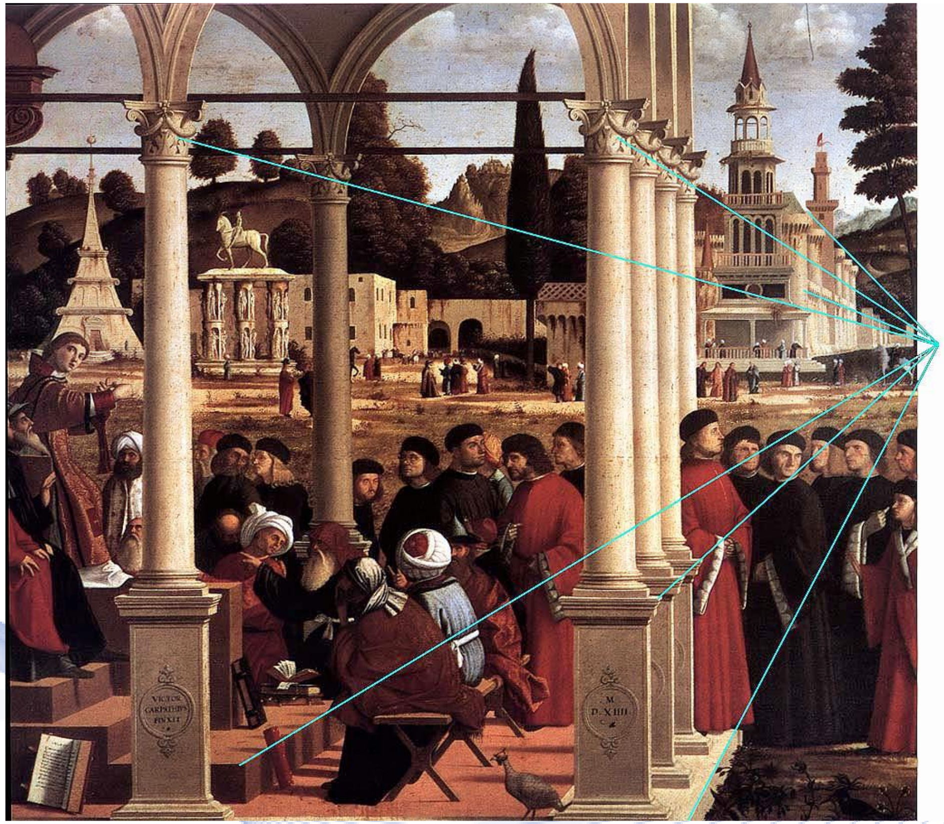
Vanishing points

A bit of History…



a) Pompei mural of the pageant of Orestes [TYL2000]; b) Byzantine icon;

A bit of History…



a) The Disputation of St Stephen' Carpaccio (1514) [TYL2000]; b) Canaletto painting.

Projective geometry

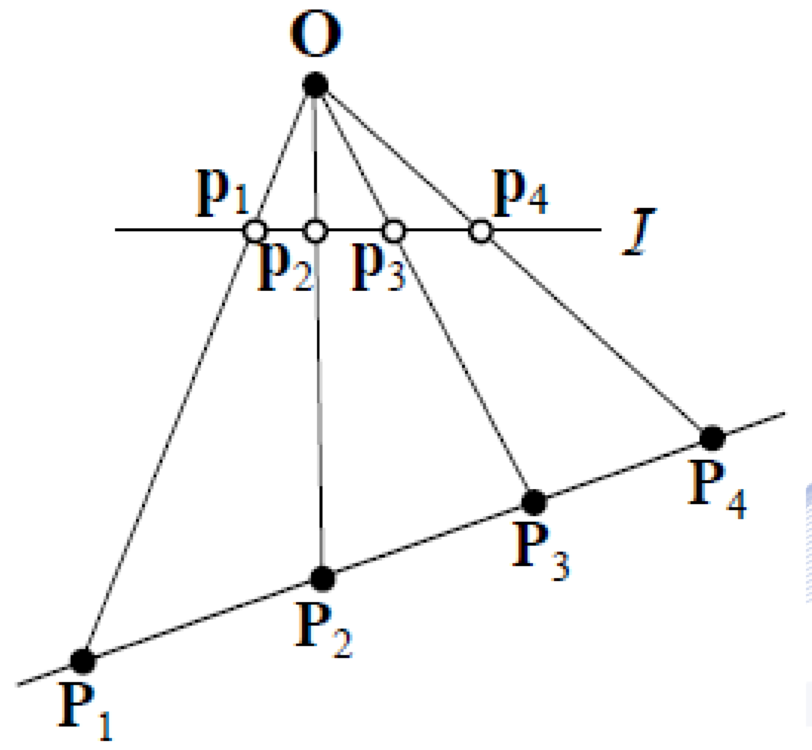
- *Projective transformation*: a product of rotations and translations in 3D ray space, forming a 3×3 matrix.
- Projection of a point $\mathbf{P}$ in $\mathbb{P}^3$ to a point $\mathbf{p}$ in $\mathbb{P}^2$:

$$\mathrm{p} = \mathcal{P}\mathrm{P}$$

- *Perspective transformation*: Special case of projective transformation.

Projective geometry

- The cross-ratio of four collinear points remains invariant under a projective transformation: $C_r(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = C_r(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4)$.



Projective geometry

- The cross-ratio invariance is useful for defining and constructing *conic sections* (curves like circles, ellipses, parabolas and hyperbolas) in the projective plane $\mathbb{P}^2$.

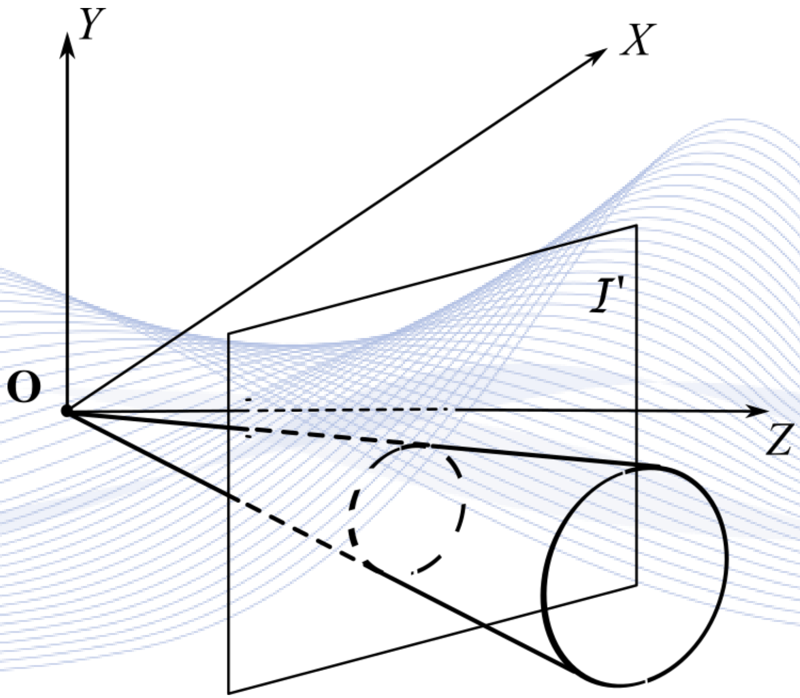
- Conic: a set of points in $\mathbb{P}^2$ which satisfy:

$$\mathbf{p}^T\mathbf{C}\mathbf{p} = 0$$

$\mathbf{C}$: a symmetric 3×3 matrix containing the equation coefficients of the conic section.

Projective geometry

- Conic sections (circles, ellipses, parabolas and hyperbolas) in the projective plane $\mathbb{P}^2$.



Q & A

Thank you very much for your attention!

More material in
http://icarus.csd.auth.gr/cvml-web-lecture-series/

Contact: Prof. I. Pitas
pitas@csd.auth.gr