Mathematical preliminaries for computer vision and deep learning

Contributor: Ioannis Mademlis, Prof. Ioannis Pitas

Presenter: Prof. Ioannis Pitas
Aristotle University of Thessaloniki
pitas@aiia.csd.auth.gr
www.multidrone.eu
Presentation version 1.2

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Mathematical preliminaries

MultiDrone

- **Mathematical Analysis**
  - Functions
  - Differentiation
  - Fourier transform
- Vector calculus
- 3D geometric transformations
- Projective geometry

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

1D, 2D, 3D analog signals/functions

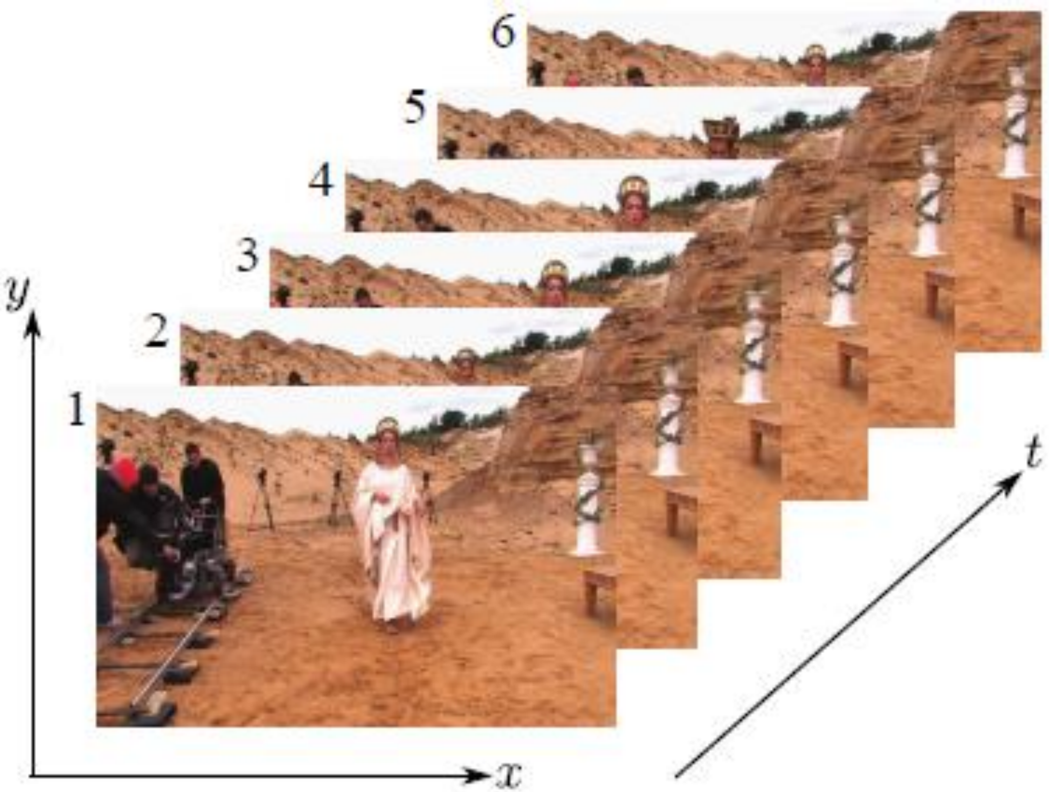
- 1D signals of the form $f(t)\colon \mathbb{R} \rightarrow \mathbb{R}$
  - *Speech, music*

- 2D signals of the form $f(x, y\,)\colon \mathbb{R}^2 \rightarrow \mathbb{R}$
  - *Greyscale images*

- 3D signals of the form $f(x, y, z)\colon \mathbb{R}^3 \rightarrow \mathbb{R}$
  - Video signals $f(x, y, t)\colon \mathbb{R}^3 \rightarrow \mathbb{R}$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D data types: video signal $f(x,y,t)$



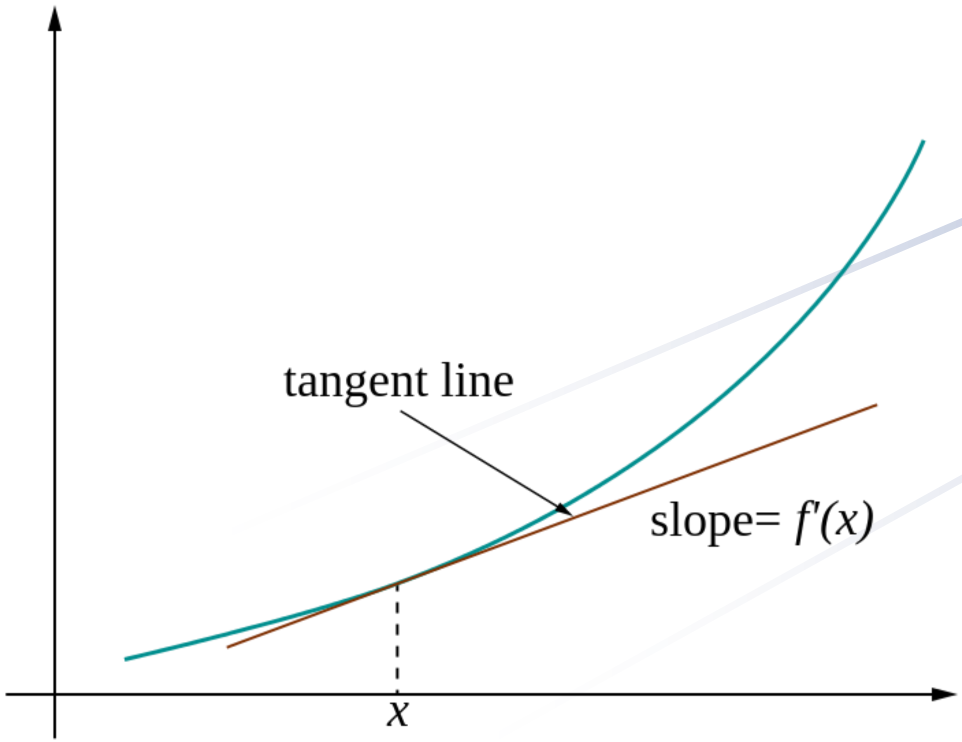
This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

1D, 2D, 3D discrete signals

- 1D signals of the form $f(n): \mathbb{Z} \rightarrow \mathbb{R}$
  - *Digital speech, music*
- 2D signals of the form $f(i,j): \mathbb{Z}^2 \rightarrow \mathbb{R}$
  - *Digital greyscale images*
- 3D signals of the
  - Volumetric images $f(i,j,k): \mathbb{Z}^3 \rightarrow \mathbb{R}$
  - Digital video signals $f(i,j,k): \mathbb{Z}^3 \rightarrow \mathbb{R}$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

1D Differentiation



- The derivative of a function at a specific point is the rate of change of the output with respect to the input.

- For 1D continuous functions, it is the slope of the tangent line to the function graph at that point:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Numerical Differentiation

- There are algorithms for approximate differentiation, using various function values.

- For instance, the slope of a nearby secant line through the points $(x-h, f(x-h))$ and $(x+h, f(x+h))$ can be employed (for small h):

$$\frac{f(x+h)-f(x-h)}{2h}$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Partial Differentiation

- For functions of two variables $f(x, y)$, the partial derivatives $\partial f/\partial x, \partial f/\partial y$ can be computed with respect to each variable.

- The grad of a function is given by the vector:

$$\boldsymbol{\nabla} f = [\partial f/\partial x, \partial f/\partial y]^T.$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Partial Differentiation

- For functions of many variables $f(\boldsymbol{x})$, $\boldsymbol{x} = [x_1, x_2, \dots, x_n]^{\mathrm{T}}$, the partial derivatives $\frac{\partial f}{\partial x_i}, i = 1, \dots, n$ can be computed with respect to each variable.

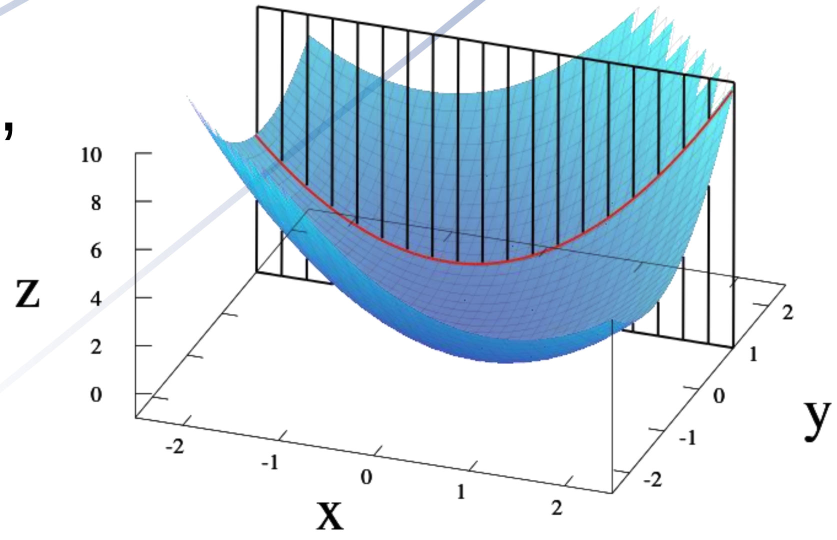
- The grad of a function is given by the vector:

$$\boldsymbol{\nabla} f = [\partial f / \partial x_1, \dots, \partial f / \partial x_n]^T.$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Partial Differentiation

- The partial derivatives give the slope of the function graph at a specific point, along the directions parallel to the coordinate axes.
- At maxima/minima, saddle points, $\nabla f = \mathbf{0}$.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Steepest Gradient Descent

- If function $f(\mathbf{x})$ is defined and differentiable in a neighborhood of a point $\mathbf{x}_t$, then $f(\mathbf{x})$ decreases fastest, going from $\mathbf{x}_t$ to $\mathbf{x}_{t+1}$ following the direction of the negative gradient of $f(\mathbf{x})$ at $\mathbf{x}_t$:

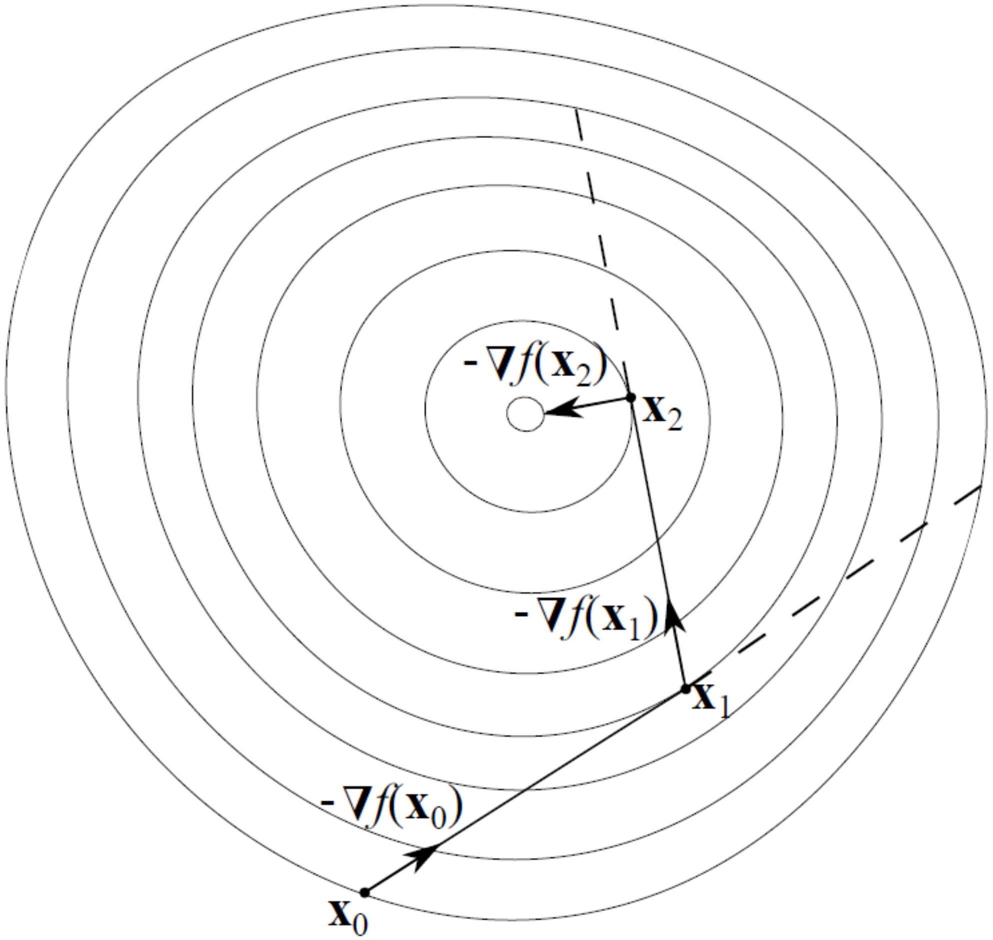
$$\mathbf{x}_{t+1} = \mathbf{x}_t - a\boldsymbol{\nabla} f(\mathbf{x}_t)$$

a: the step used to update the vector $\mathbf{x}_{t+1}$ at each iteration t.

- $f(\mathbf{x}_{t+1}) \leq f(\mathbf{x}_t)$ and sequence $\mathbf{x}_t$ converges to a local minimum of $f(\mathbf{x}_t)$.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Steepest Gradient Descent



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Fourier transform

- *1D Fourier transform:*

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx \qquad f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

- *2D Fourier transform:*

$$F_\alpha(\Omega_x, \Omega_y) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_\alpha(x, y) \exp(-i\Omega_x x - i\Omega_y y)\, dx dy$$

$$f_\alpha(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_\alpha(\Omega_x, \Omega_y) \exp(i\Omega_x x + i\Omega_y y)\, d\Omega_x d\Omega_y$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

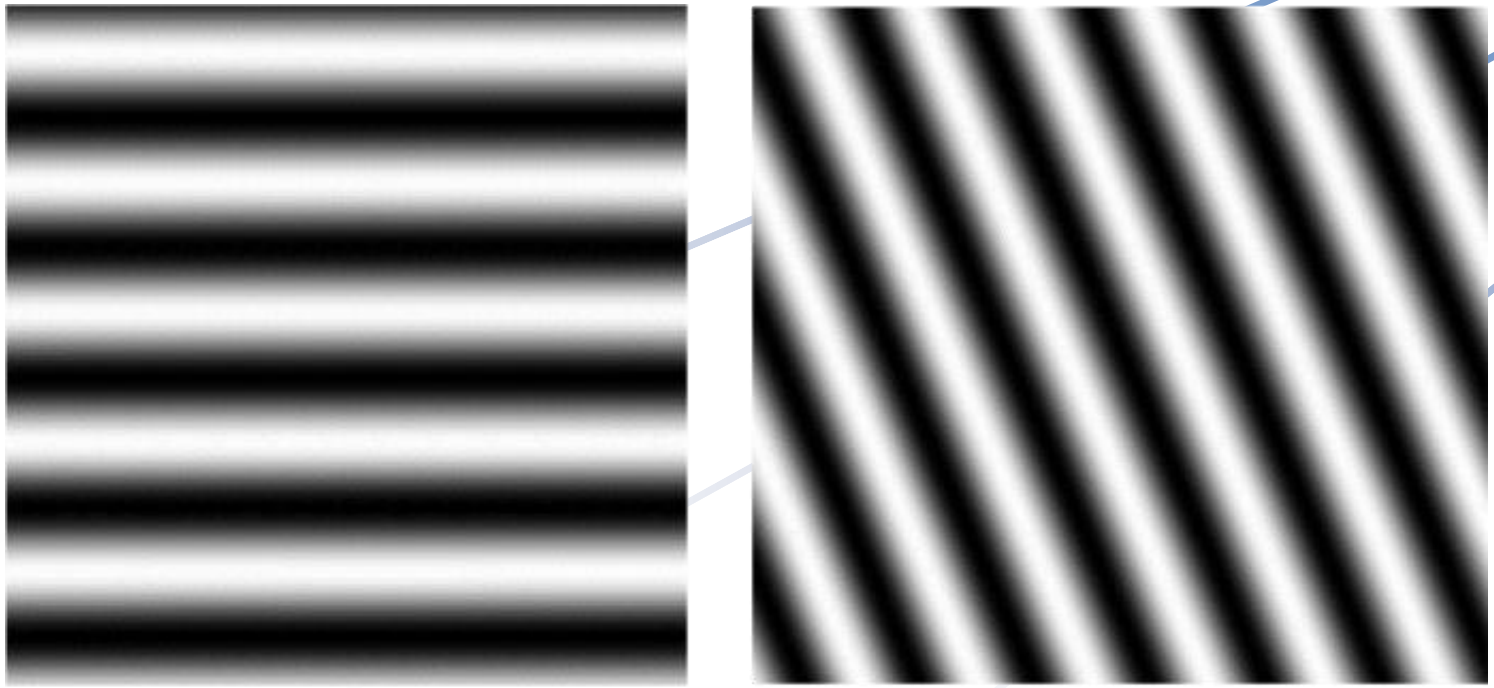
Fourier transform

- F_x, F_y: 2D spatial frequencies representing how rapidly image luminance or chrominance changes on the image plane:
  - in *cycles per unit length* along a given axis,
  - in *cycles per meter (cpm)* in the metric measure system.

- $\Omega_x = 2\pi F_x,\ \Omega_y = 2\pi F_y$: respective angular frequencies.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Spatial image content



$$f(x, y) = \sin(20\pi x + 8\pi y)$$

$$(\Omega_x = 20\pi, \Omega_x = 8\pi)$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Mathematical preliminaries

MultiDrone

- Mathematical Analysis
  - Functions
  - Differentiation
  - Fourier transform
- **Vector calculus**
- 3D geometric transformations
- Projective geometry

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Vector calculus

- Vectors of the form $\mathbf{P} = [\mathrm{X}, \mathrm{Y}, \mathrm{Z}]^T$ define point positions in the 3D space $\mathbb{R}^3$.

- Vectors of the form $\mathbf{p} = [x, y]^T$ define point positions in the 2D space $\mathbb{R}^2$.

- The right-hand thumb rule is typically followed when defining the axis system (X, Y, Z) in $\mathbb{R}^3$.

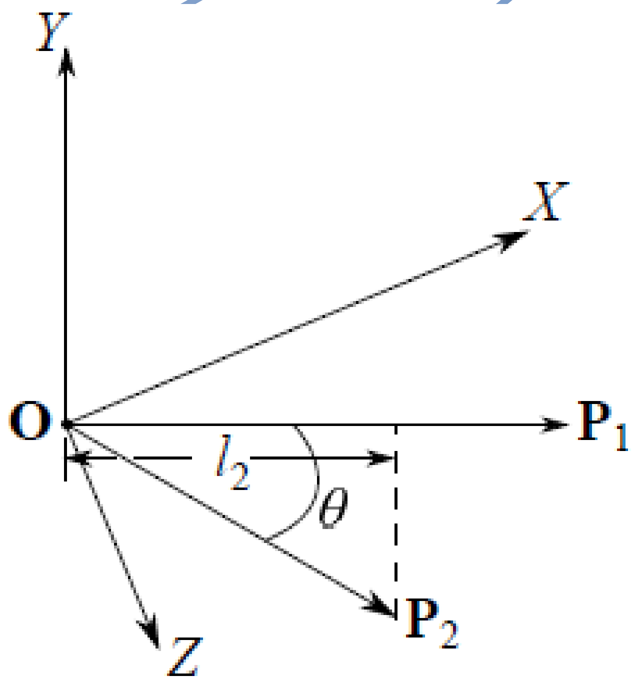
This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Vector calculus

- *Inner vector product* or *dot product* in $\mathbb{R}^3$:

$$\mathbf{P}_1^T\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1^T = X_1X_2 + Y_1Y_2 + Z_1Z_2$$
$$= \|\mathbf{P}_1\| \cdot \|\mathbf{P}_2\| \cos\theta.$$

θ: the angle formed by the two vectors.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Vector calculus

- Properties of the inner vector product:
  - It is a scalar value.
  - Its value is equal to the product of the length of one vector $\|\mathbf{P}_1\|$ and the length of the projection of the second vector $l_2 = \|\mathbf{P}_2\| \cos\theta$ on the first one.
  - It is maximal for collinear vectors, i.e., $\theta = 0$, thus $\mathbf{P}_1{}^T\mathbf{P}_2 = \|\mathbf{P}_1\| \cdot \|\mathbf{P}_2\|$.
  - For unit vectors $\|\mathbf{P}_1\| = \|\mathbf{P}_2\| = 1$, $\mathbf{P}_1{}^T\mathbf{P}_2 = \cos\theta$.

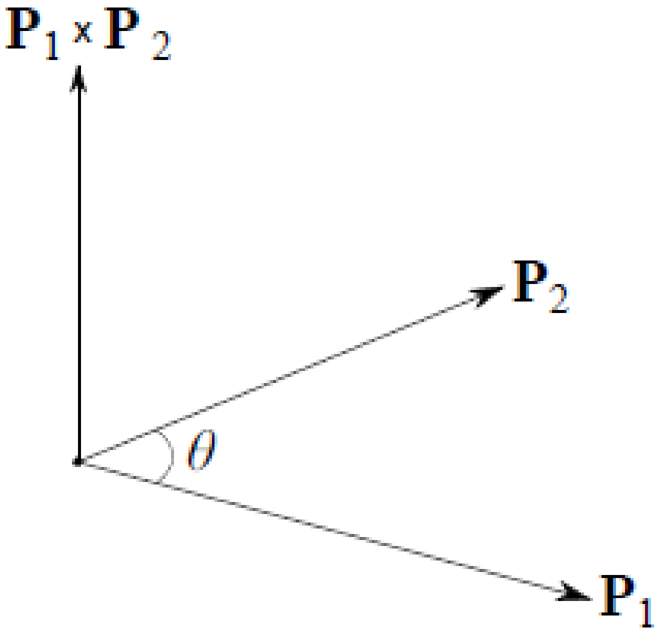
This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Vector calculus

- *Cross vector product* $\mathbf{P}_1 \times \mathbf{P}_2$ in $\mathbb{R}^3$:

$$\mathrm{P}_1 \times \mathrm{P}_2 = (Y_1 Z_2 - Z_1 Y_2)\mathrm{i} + (Z_1 X_2 - X_1 Z_2)\mathrm{j} + (X_1 Y_2 - Y_1 X_2)\mathrm{k}$$

$\mathbf{i}, \mathbf{j}, \mathbf{k}$: the standard basis vectors of $\mathbb{R}^3$.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Vector calculus

- Properties of the cross vector product:
  - It is a vector perpendicular to both $\mathbf{P}_1$ and $\mathbf{P}_2$.
  - Its length is equal to the area of the parallelogram spanned by the two vectors $\|\mathbf{P}_1\| \cdot \|\mathbf{P}_2\| \sin\theta$, where θ the angle formed by the two vectors.
  - It may also be expressed as a matrix multiplication:

$$\mathrm{P}_1 \times \mathrm{P}_2 = [\mathrm{P}_1]_\times \mathrm{P}_2 = \begin{bmatrix} 0 & -Z_1 & Y_1 \\ Z_1 & 0 & -X_1 \\ -Y_1 & X_1 & 0 \end{bmatrix} \mathrm{P}_2$$

with the 3×3 cross product matrix of $\mathbf{P}_1$ having rank 2.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Vector calculus

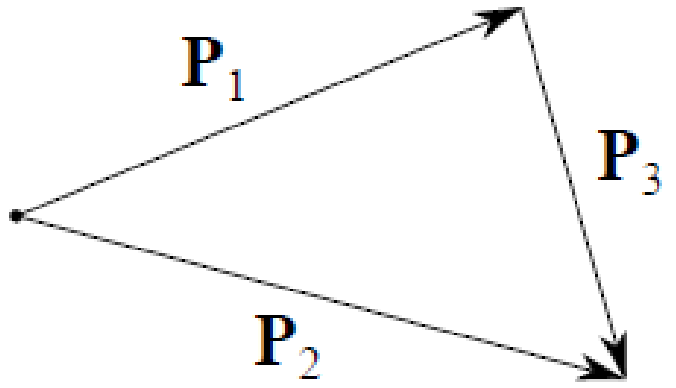
- *Scalar triple product* of three vectors $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$:

$$\mathbf{P}_3{}^T(\mathbf{P}_1 \times \mathbf{P}_2)$$

- It is the inner product of one of the vectors with the cross product of the other two.
- If the vectors are coplanar, then:

$$\mathbf{P}_3{}^T(\mathbf{P}_1 \times \mathbf{P}_2) = 0.$$

- It expresses the volume of the parallelepiped defined by the three vectors $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Mathematical preliminaries

MultiDrone

- Mathematical Analysis
  - Functions
  - Differentiation
  - Fourier transform
- Vector calculus
- **3D geometric transformations**
- Projective geometry

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

- 3D solid object motions: superposition of a 3D rotation and a 3D translation:

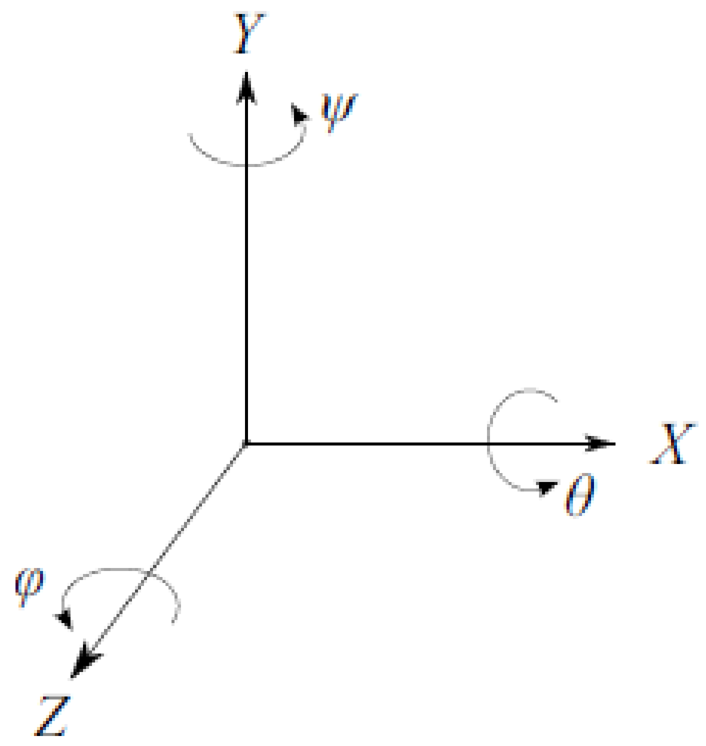
$$\mathbf{X}' = \mathbf{R}\mathbf{X} + \mathbf{T}$$

- $\mathbf{X} = [X, Y, Z]^T$, $\mathbf{X}' = [X', Y', Z']^T$: the coordinates of a solid object point at time instances t and t'.
- $\mathbf{R}$ is a 3×3 rotation matrix, which can be defined by either:
  - The Euler rotation angles about X, Y, Z axes (in Cartesian coordinates)
  - a unitary rotation axis and a rotation angle about this axis.
- $\mathbf{T} = \left[T_x, T_y, T_z\right]^T$: a 3D translation vector.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

- An arbitrary rotation in the 3D space can be represented by the Euler rotation angles θ, ψ, ϕ about the X, Y, Z axes.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

- Matrix representation of clockwise rotation about each X, Y, Z axis:

$$\mathbf{R} = \mathbf{R}_z\mathbf{R}_y\mathbf{R}_x = \begin{bmatrix} \cos\phi\cos\psi & \cos\phi\sin\psi\sin\theta - \sin\phi\cos\theta & \cos\phi\sin\psi\cos\theta + \sin\phi\sin\theta \\ \sin\phi\cos\psi & \sin\phi\sin\psi\sin\theta + \cos\phi\cos\theta & \sin\phi\sin\psi\cos\theta - \cos\phi\sin\theta \\ -\sin\psi & \cos\psi\sin\theta & \cos\psi\cos\theta \end{bmatrix}$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad \mathbf{R}_y = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \quad \mathbf{R}_z = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The order of matrices in this equation does matter.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

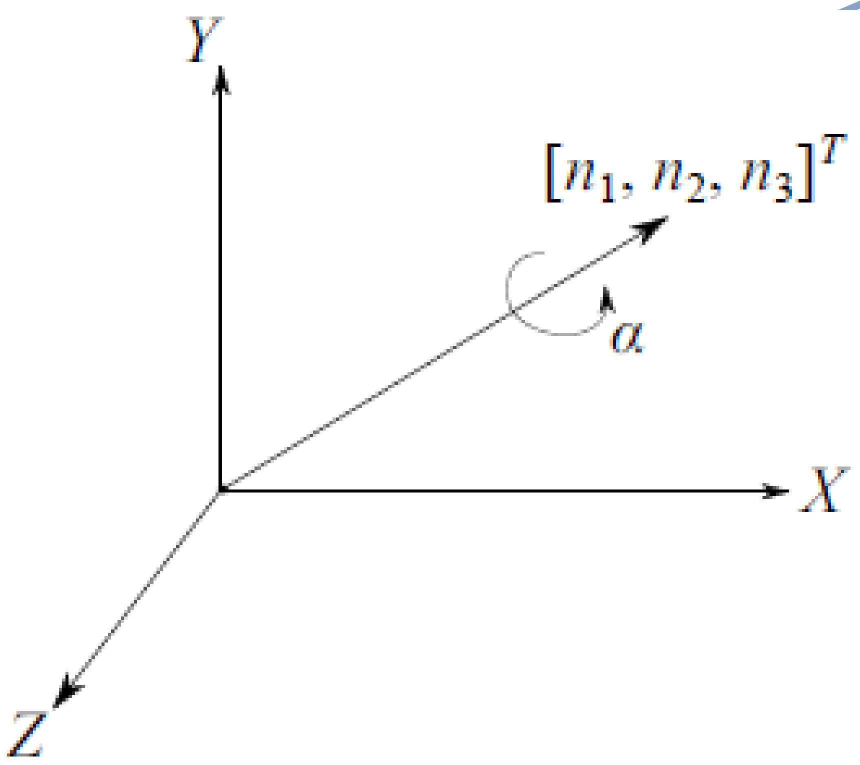
3D geometric transformations

- Orthonormal matrix $\mathbf{R}$ satisfies: $\mathbf{R}^T = \mathbf{R}^{-1}, det(\mathbf{R}) = \pm 1$.
- For infinitesimal rotations of a 3D point $\theta \approx \Delta\theta \approx 0, \varphi \approx \Delta\varphi \approx 0, \psi \approx \Delta\psi \approx 0$ approximations $\cos\Delta\varphi \approx 1$ and $\sin\Delta\varphi \approx \Delta\varphi \approx 0$ can be employed.
- Then, matrix multiplication order is irrelevant and $\mathbf{R}$ takes the following form:

$$\mathrm{R} = \begin{bmatrix} 1 & -\Delta\phi & \Delta\psi \\ \Delta\phi & 1 & -\Delta\theta \\ -\Delta\psi & \Delta\theta & 1 \end{bmatrix}$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

- Matrix representation of a 3D solid object rotation by an angle α about arbitrary axis passing through the origin and determined by the unit vector orientation vector $\mathbf{n} = [n_1, n_2, n_3]^T$: $\mathrm{R} = \cos\alpha\mathbf{I} + \sin\alpha[\mathbf{n}]_\times + (1-\cos\alpha)\mathbf{n}\otimes\mathbf{n} =$

$$= \begin{bmatrix} \cos\alpha & 0 & 0 \\ 0 & \cos\alpha & 0 \\ 0 & 0 & \cos\alpha \end{bmatrix} + \begin{bmatrix} 0 & -n_3\sin\alpha & n_2\sin\alpha \\ n_3\sin\alpha & 0 & -n_1\sin\alpha \\ -n_2\sin\alpha & n_1\sin\alpha & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} n_1^2(1-\cos\alpha) & n_1n_2(1-\cos\alpha) & n_1n_3(1-\cos\alpha) \\ n_2n_1(1-\cos\alpha) & n_2^2(1-\cos\alpha) & n_2n_3(1-\cos\alpha) \\ n_3n_1(1-\cos\alpha) & n_3n_2(1-\cos\alpha) & n_3^2(1-\cos\alpha) \end{bmatrix} =$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

$$= \begin{bmatrix} n_1^2 + (1 - n_1^2)\cos\alpha & n_1 n_2(1 - \cos\alpha) - n_3 \sin\alpha & n_1 n_3(1 - \cos\alpha) + n_2 \sin\alpha \\ n_1 n_2(1 - \cos\alpha) + n_3 \sin\alpha & n_2^2 + (1 - n_2^2)\cos\alpha & n_2 n_3(1 - \cos\alpha) - n_1 \sin\alpha \\ n_1 n_3(1 - \cos\alpha) - n_2 \sin\alpha & n_2 n_3(1 - \cos\alpha) + n_1 \sin\alpha & n_3^2 + (1 - n_3^2)\cos\alpha \end{bmatrix}$$

where:

$$\mathbf{n} \otimes \mathbf{n} = \begin{bmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 n_3 & n_3^2 \end{bmatrix}$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

- For an infinitesimal rotation angle $\Delta\alpha \approx 0$:

$$\mathrm{R} = \begin{bmatrix} 1 & -n_3\Delta\alpha & n_2\Delta\alpha \\ n_3\Delta\alpha & 1 & -n_1\Delta\alpha \\ -n_2\Delta\alpha & n_1\Delta\alpha & 1 \end{bmatrix}$$

- The infinitesimal rotation assumption holds when object motion is relatively slow and or the time interval is very small.
  - It can be used in video analysis if we take into account the relatively short time interval between video frames.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

MultiDrone

- 3D rotation can also be represented by *quaternions* that are extensions of complex numbers:

$$\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

q_0, q_1, q_2, q_3 are real numbers and:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$$

- Unit quaternion $\mathbf{q}_R = [q_0\ q_1\ q_2\ q_3]^T$. It satisfies:

$$q_0{}^2 + q_1{}^2 + q_2{}^2 + q_3{}^2 = 1.$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

3D geometric transformations

- Rotation by an angle α around a unit vector $[n_1, n_2, n_3]^T$:

$$\mathbf{q} = \begin{bmatrix} n_1 \sin\frac{\alpha}{2} & n_2 \sin\frac{\alpha}{2} & n_3 \sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{bmatrix}^T$$

- Rotation matrix $\mathbf{B}$ corresponding to a certain quaternion:

$$\mathbf{R} = \begin{bmatrix} q_0^2 - q_1^2 - q_2^2 - q_3^2 & 2(q_0 q_1 + q_2 q_3) & 2(q_0 q_2 - q_1 q_3) \\ 2(q_0 q_1 - q_2 q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1 q_2 + q_0 q_3) \\ 2(q_0 q_2 + q_1 q_3) & 2(q_1 q_2 - q_0 q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{bmatrix}$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Mathematical preliminaries

MultiDrone

- Mathematical Analysis
  - Functions
  - Differentiation
  - Fourier transform
- Vector calculus
- 3D geometric transformations
- **Projective geometry**

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- *Homogeneous coordinates* are basic concept in projective geometry.
- Assuming an Euclidean plane $\mathbb{R}^2$ and a Cartesian coordinate system defined on it, $\mathbf{p} = [x, y]^T$, we assign the ordered 3-tuple $[x_h, y_h, a]^T$ to $\mathbf{p}$, where $a \in \mathbb{R}, a \neq 0, x = x_h/a\, , y = y_h/a$.
- The homogeneous coordinates $[x_h, y_h, a]^T$ define the projective plane $\mathbb{P}^2$.
    - Exactly one Euclidean point corresponds to each 3-tuple $[x_h, y_h, a]^T$.
    - But both $[x_h, y_h, a]^T$ and $[\lambda x_h, \lambda y_h, \lambda a]^T$ 3-tuples correspond to the same Euclidean coordinates $[x, y]^T$, where $\lambda \in \mathbb{R}, \lambda \neq 0$.
    - Canonical form $[x_h, y_h, 1]^T$???.
    - The projective plane can be defined over the real or the complex field $x_h, y_h, a \in \mathbb{C}$???.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

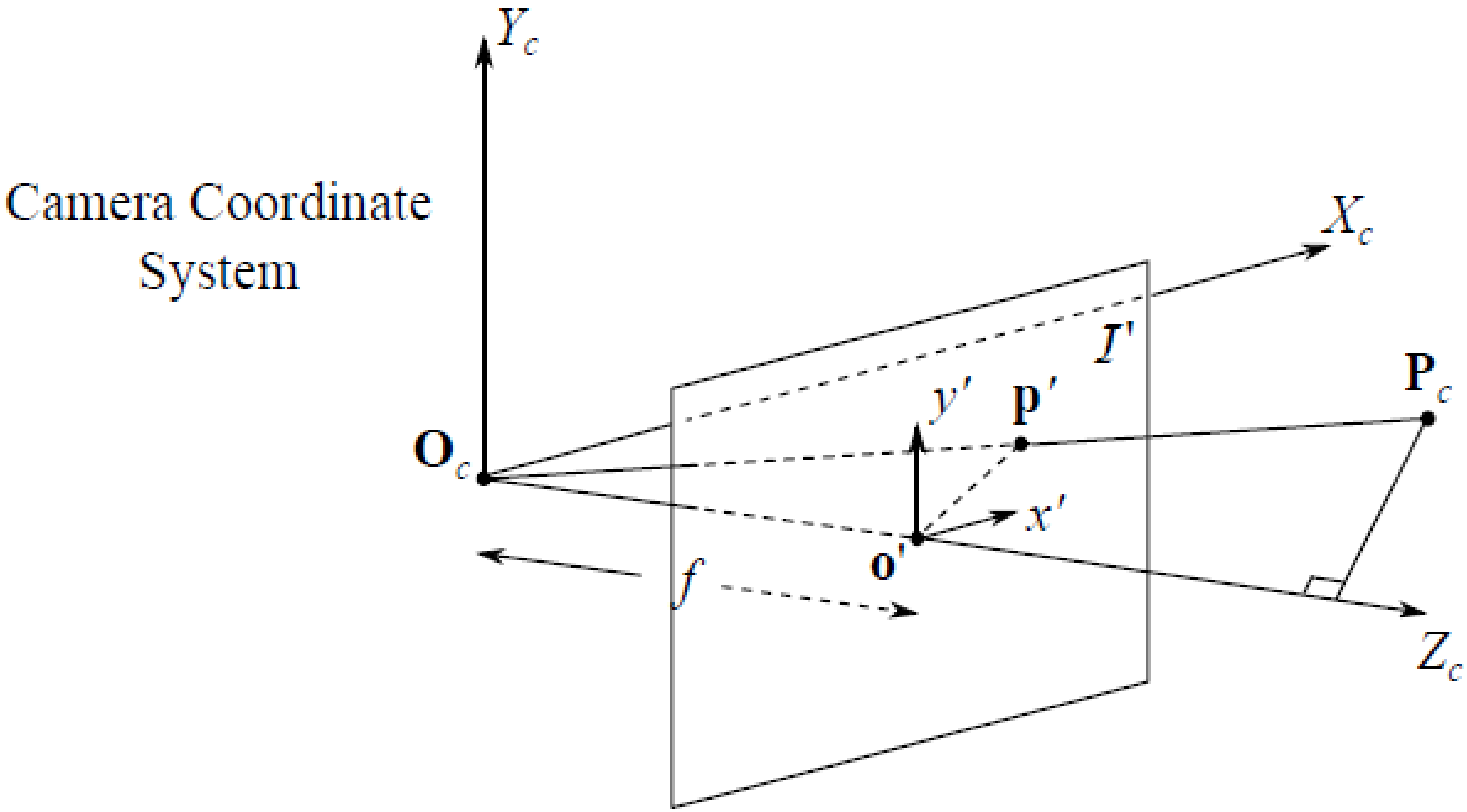
- Assuming a 3D Euclidean space $\mathbb{R}^3$ and a Cartesian coordinate system defined on it, $\mathbf{P} = [X, Y, Z]^T$, we assign the ordered 4-tuple $[X_h, Y_h, Z_h, a]^T$ to $\mathbf{P}$, where $a \in \mathbb{R}, a \neq 0, X = X_h/a\,, Y = Y_h/a\,, Z = Z_h/a$.

- The homogeneous coordinates $[X_h, Y_h, Z_h, a]^T$ define the projective space $\mathbb{P}^3$.
  - Exactly one Euclidean point corresponds to each 4-tuple $[X_h, Y_h, Z_h, a]^T$.
  - But both 4-tuples $[X_h, Y_h, Z_h, a]^T$ and $[\lambda X_h, \lambda Y_h, \lambda Z_h, \lambda a]^T$ correspond to the same 3D Euclidean coordinates $[X, Y, Z]^T$, where $\lambda \in \mathbb{R}, \lambda \neq 0$.
  - $[X_h, Y_h, Z_h, 1]^T \quad ???$.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

Pinhole camera: projection of the 3D world $\mathbb{R}^3$ on an image plane $\mathbb{R}^2$.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- Difference from Euclidean geometry: it allows for a wider range of transformations than just rotations and translations.
  - *Perspective (or central) projections:*
    - They have a projection center $\mathbf{O}$.
    - Line segment lengths, angles or parallelism are not necessarily preserved, leading to shape distortions.
    - Property invariable during projection from the *projective space* $\mathbb{P}^3$ to the projective plane $\mathbb{P}^2$: *incidence*
      - collinear points remain collinear.
      - straight lines are mapped onto straight lines.
    - Enable the viewer to perceive depth differences between items of the projected 3D scene from the resulting 2D image.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- Fundamental property of perspective projection:
  - Every image point lies on a 3D line passing through its corresponding scene point in world space and a common *center of projection.*
- The 2D projective plane $\mathbb{P}^2$ can be considered as contained within a 3D Euclidean "ray space", containing 3D straight lines passing through origin $\mathbf{O}$ and point $\mathbf{p} = [x, y]^T$, corresponding to the ordered 3-tuple $[x_h, y_h, a]^T$.
  - Thus, the 2D projective plane can describe a sequence of one or more perspective projections from a plane in 3D space to the image plane.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- An infinite number of 3-tuples $\mathbf{p}_{\mathrm{H}} = [x_h, y_h, a]^T$ corresponds to each point on the Euclidean plane, hence:
  - The homogeneous coordinates of point $\mathbf{p} = [x, y]^T$ are not uniquely defined.

- Dropping the constraint $a \neq 0$ we can have *points at infinity* with homogeneous coordinates $[x_h, y_h, 0]^T$, where ${x_h}^2 + {y_h}^2 \neq 0$, which lie on the *line at infinity* $[0,0,a]^T$ on $\mathbb{P}^2$.
- Canonical form of line at infinity $[0,0,1]^T$.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- A (straight) *line* in $\mathbb{P}^2$ is defined by its parameter vector $\mathbf{u} = [a, b, c]^T$, satisfying $\mathbf{p}^T\mathbf{u} = ax_h + by_h + ca = 0$, where $\mathbf{p} = [x_h, y_h, a]^T$.
- Line equation coefficients $\mathbf{u}$: they are the vector entries of homogeneous coordinates of the corresponding projective line.
- Equation of the line joining points $\mathbf{p}_1 = [x_{h1}, y_{h1}, a_1]^T, \mathbf{p}_2 = [x_{h2}, y_{h2}, a_2]^T$ in $\mathbb{P}^2$: $\mathbf{u} = \mathbf{p}_1 \times \mathbf{p}_2$.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- *Intersection point* of two lines $\mathbf{u}_1 = [a_1, b_1, c_1]^T, \mathbf{u}_2 = [a_2, b_2, c_2]^T$:

$$\mathbf{p} = \mathbf{u}_1 \times \mathbf{u}_2$$

- A *line at infinity* can be described by the homogeneous coordinate vector $[0,0,1]^T$.
    - It is defined by the plane of rays which pass through the center of projection and are parallel to the image plane.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- Three homogeneous coordinate points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ lie on the same line if $det(\mathbf{P}) = 0$, where $\mathbf{P} = [\mathbf{p}_1|\mathbf{p}_2|\mathbf{p}_3]$.
- Three homogeneous coordinate lines $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ intersect at the same point if $det(\mathbf{U}) = 0$, where $\mathbf{U} = [\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_3]$.
- A point $\mathbf{p}$ lies on a line $\mathbf{u}$, if and only if $\mathbf{p}^T\mathbf{u} = 0$.
- *Duality* principle: lines and points are interchangeable in equations on $\mathbb{P}^2$.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- The addition of the line at infinity $[0,0,1]^T$ to the Euclidean plane $[x, y]^T$ augments it to a projective plane $[x, y, 1]^T$.
  - Infinite homogeneous coordinates correspond to the same Euclidean point, for different values of λ.
  - A homogeneous point in $\mathbb{P}^2$ corresponds to an entire ray passing through the origin in the associated 3D Euclidean ray space.
  - Each homogeneous equation of the form $a_1x_h + a_2y_h + a_0a = 0$ with ${a_1}^2 + {a_2}^2 + {a_0}^2 \neq 0$ represents a line.
    - for $a = 0$: line at infinity.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- Points at infinity: the points of intersection on $\mathbb{P}^2$ between parallel lines of $\mathbb{R}^2$.
  - No real line parallelism exists in the projective plane.
- Each point at infinity is associated with a specific direction given by the slope of the parallel $\mathbb{R}^2$ lines intersecting at it.
- Linear transformation on homogeneous coordinates preserving the property of incidence:
  - Rotation and translation of the 3D ray space coordinate system.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

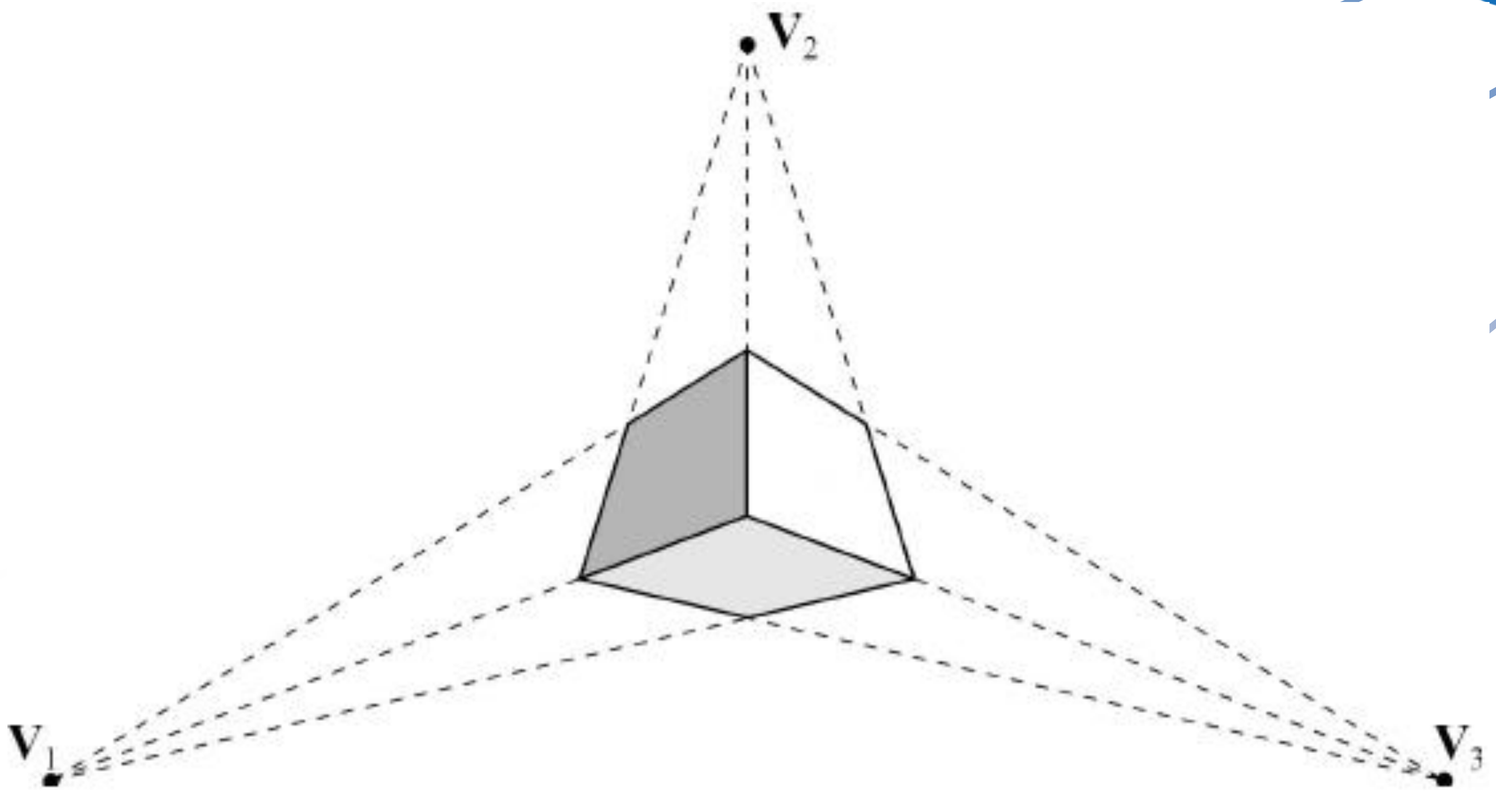
- *Vanishing points*: the points of intersection between projected lines corresponding to parallel lines in the Euclidean space.

- Points at infinity in $\mathbb{P}^2$ are mapped to vanishing points in $\mathbb{R}^2$. ???

- Every point of the plane at infinity maps onto an image point in $\mathbb{P}^2$, during perspective projection.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry



Vanishing points

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- *Projective transformation*: a product of rotations and translations in 3D ray space, forming a 3×3 matrix.
- Projection of a point $\mathbf{P}$ in $\mathbb{P}^3$ to a point $\mathbf{p}$ in $\mathbb{P}^2$:

$$\mathrm{p} = \mathcal{P}\mathrm{P}$$

- *Perspective transformation*: Special case of projective transformation ???

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- *Affine transformation*: ???
- Projection of a point $\mathbf{P}$ in $\mathbb{P}^3$ to a point $\mathbf{p}$ in $\mathbb{P}^2$:

$$\mathrm{p} = \mathcal{P}\mathrm{P}$$

- *Similarity transformation*: ???

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- *Affine plane*: intermediate construction, in-between the Euclidean and the projective plane:
  - It contains exactly the same points as the Euclidean plane.
  - Parallelism and ratios of distances between collinear points are preserved during affine transformations.
  - It allows uniform and non-uniform scaling and shear transformations.
- *Similarity plane:*
  - It allows only rotation, translation and uniform scaling.
  - It preserves almost all geometrical properties.
  - It does **not** preserve the line segment length.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- Geometrical transformations, from the most to the least property-preserving ones:
  - Euclidean
  - Similarity
  - Affine
  - Projective.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- *Cross-ratio (or anharmonic ratio)* C_r: ratio of ratios of distances between collinear points.
  - It is a geometric property invariant under a projective transformation.

- For four points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ in $\mathbb{P}^2$:

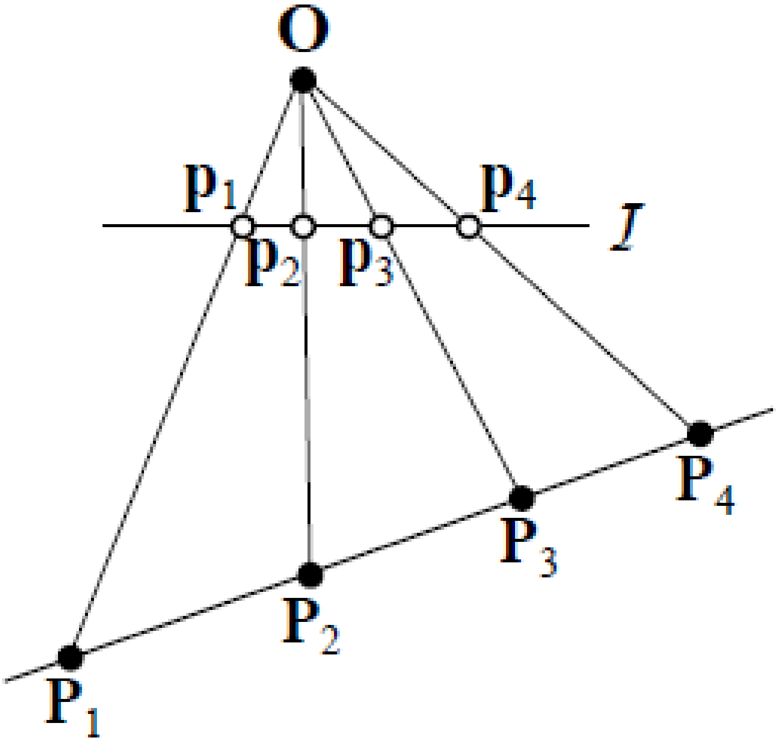
$$C_r(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{\Delta_{13}\Delta_{24}}{\Delta_{14}\Delta_{23}}$$

Δ_{ij}: the Euclidean distance between points $\mathbf{p}_i$ and $\mathbf{p}_j$.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- The cross-ratio of four collinear points remains invariant under a projective transformation: $C_r(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = C_r(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3, \mathbf{P}_4)$.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- The cross-ratio invariance is useful for defining and constructing *conic sections* (curves like circles, ellipses, parabolas and hyperbolas) in the projective plane $\mathbb{P}^2$.

- Conic: a set of points in $\mathbb{P}^2$ which satisfy:

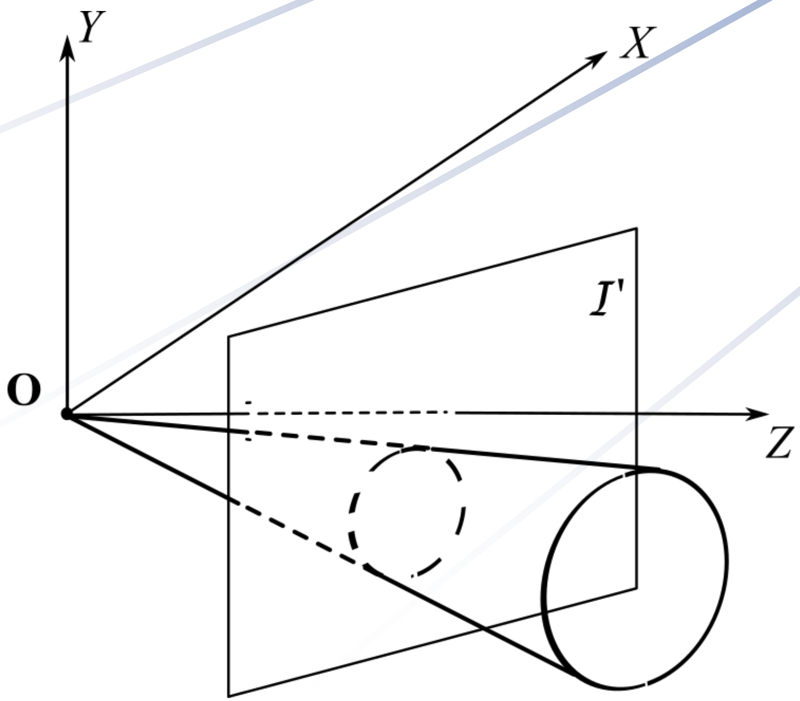
$$\mathbf{p}^T \mathbf{C} \mathbf{p} = 0$$

$\mathbf{C}$: a symmetric 3×3 matrix containing the equation coefficients of the conic section.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- Conic sections (circles, ellipses, parabolas and hyperbolas) in the projective plane $\mathbb{P}^2$.



This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- All conic sections are projectively equivalent.
- Each type of conic section can be projectively mapped to any other type of conic section.
- All circles on the complex projective plane intersect the line at infinity at two fixed points, the *absolute* or *circular points*, having complex coordinates $\mathbf{i} = [1, i, 0]^T$ and $\mathbf{j} = [1, -i, 0]^T$.
  - The absolute points are invariant under similarity and Euclidean transformations.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- In the $\mathbb{P}^3$ projective space:
  - points are dual with planes and
  - lines are represented as the intersection of two planes.
- Projection of a point $\mathbf{P}$ in $\mathbb{P}^3$ to a point $\mathbf{p}$ in $\mathbb{P}^2$:

$$\mathrm{p} = \mathcal{P}\mathrm{P}$$

- $\mathbb{P}^3$ can be considered as embedded into a 4D 'ray space', with each 4D ray passing through the origin and intersecting with $\mathbb{P}^3$ at a single point.
- *Plane at infinity*: defined by $a = 0$ $([X_h, Y_h, Z_h, 0]^T)$ and containing all points at infinity.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- *Quadric*
  - a surface defined as the set of points $\mathbf{P}$ satisfying the equation $\mathbf{P}^T\mathbf{Q}\mathbf{P} = 0$.
    - $\mathbf{Q}$ is a 4×4 symmetric quadric coefficient matrix.
  - They are a 3D generalization of conics.
  - Their intersection with a plane is always a conic.
  - Each of them can be fully represented by its corresponding coefficient matrix.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- In homogeneous coordinates, all spheres intersect the plane at infinity in a curve having points $[X_h, Y_h, Z_h, 0]^T$, where ${X_h}^2 + {Y_h}^2 + {Z_h}^2 = 0$ and $a = 0$.
- *Absolute conic* $\boldsymbol{\omega}$: a conic lying on the plane at infinity and consisting solely of complex points: $\mathbf{P}^T \boldsymbol{\omega} \mathbf{P} = 0$.
- $\boldsymbol{\omega}$ is a 3×3 matrix and $\mathbf{P} \in [X_h, Y_h, Z_h, 0]^T$.???
    - The identification of its projection on the image plane is equivalent to the specification of the angle between rays when only their image projections are known.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- $\boldsymbol{\omega}$ gives a way to define angles in projective space by defining an inner product:

$$\cos\theta = \frac{(\mathbf{u}_1^T \omega \mathbf{u}_2)}{\sqrt{(\mathbf{u}_1^T \omega \mathbf{u}_1)(\mathbf{u}_2^T \omega \mathbf{u}_2)}}$$

θ: the angle formed by two lines $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{P}^3$???.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- Rotation in the homogeneous coordinates: $\mathbf{P'}_h = \tilde{\mathbf{R}}\mathbf{P}_h$
- Translation in the homogeneous coordinates : $\mathbf{P'}_h = \tilde{\mathbf{T}}\mathbf{P}_h$
- Scaling in the homogeneous coordinates : $\mathbf{P'}_h = \tilde{\mathbf{S}}\mathbf{P}_h$

$$\tilde{\mathrm{R}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \tilde{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 & T_1 \\ 0 & 1 & 0 & T_2 \\ 0 & 0 & 1 & T_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \tilde{\mathrm{S}} = \begin{bmatrix} S_1 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- r_{ij}: the elements of the rotation matrix $\mathbf{R}$ in the Euclidean space.

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Projective geometry

- 3D rotation followed by translation:

$$\mathrm{P}'_h = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_1 \\ r_{21} & r_{22} & r_{23} & T_2 \\ r_{31} & r_{32} & r_{33} & T_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{P}_h$$

- 3D translation followed by rotation:

$$\mathrm{P}'_h = \begin{bmatrix} r_{11} & r_{12} & r_{13} & -\mathrm{R}_1^T\mathrm{T} \\ r_{21} & r_{22} & r_{23} & -\mathrm{R}_2^T\mathrm{T} \\ r_{31} & r_{32} & r_{33} & -\mathrm{R}_3^T\mathrm{T} \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{P}_h$$

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)

Q & A

Thank you very much for your attention!

Contact: Prof. I. Pitas
pitas@aiia.csd.auth.gr
www.multidrone.eu

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 731667 (MULTIDRONE)